UNIT I : DISCRETE FOURIER TRANSFORM

Sampling Theorem, Concept of frequency in discrete-time-Signals, Summary of analysis and synthesis equations for FT and DTFT, frequency domain sampling - Discrete Fourier Transform (DFT) - deriving DFT from DTFT, properties of DFT - Periodicity, symmetry - circular convolution - Linear Filtering Using DFT - Filtering Long data sequences - overlap save and overlap add method, Fast computation of DFT - Radix-2 Decimation in Time (DIT), Fast Fourier Transform (FFT), Decimation-in-frequency (DIF) Fast Fourier Transform (FFT) Linear Filtering Using FFT.

Discrete Signals:

→ These signals take only finite amplitude levels and they are preset at discrete time Intervals.

→ The signal which is discrete in time and amplitude is called digital signal.

Eg: PCM signal, digital image signal, digital speech etc.

Discrete systems:

→ These systems process Discrete (or) digital signals.

→ These systems are made up of digital flip-flop shift registers, Counters, ALU, Shifters, etc.

Basic Elements of DSP:

A/D Converter:

→ Analog input signal is converted to digital form.

→ It determines the sampling rate and Quantization error in digitizing operation.

Digital signal processor:

→ The DSP processor receive digital signal from A/D converter.

→ Perform the operations such as amplification, attenuation filtering, spectral analysis, Feature extraction, correlation operation on digital data.

D/A Converter:

→ The processed digital signal are provide their analog form.

→ Eg: Sound, Image and video signal are required in analog form

Advantage of DSP over Analog signal processing:

(i) DSP systems are highly flexible, they can be reconfigured for some other operation by changing software program.

(ii) DSP system do not suffer from component tolerances hence they are highly accurate.

(iii) Digital storage do not suffer from noise and distortion

(iv) Mathematical operations can be implemented in digital domain.

DisAdvantage:

(i) Difficult to process high bandwidth analog signal.

(ii) DSP Systems are expensive for small application

(iii) DSP system cannot be implemented without power supply But analog circuits can be implemented with passive components.

Application of DSP:

(i) Voice and speech: Speech recognition, voice mail, Speech coding/decoding, Speaker Identification.

(ii) TeleCommunications: Cellular phones, video Conferencing, echo cancellation, data Encryption.

(iii) DSP for graphics and Imaging: 3-D and 2-D Visualization, animation, Image coding & Transmission

(V) DSP for military and Defense: Missile guidance, RF Modems, secure Communication.

(vi) Biomedical Engineering: X-ray Storage, ECG Analysis, CT Scanning Equipments, Patient monitoring systems.

Sampling:

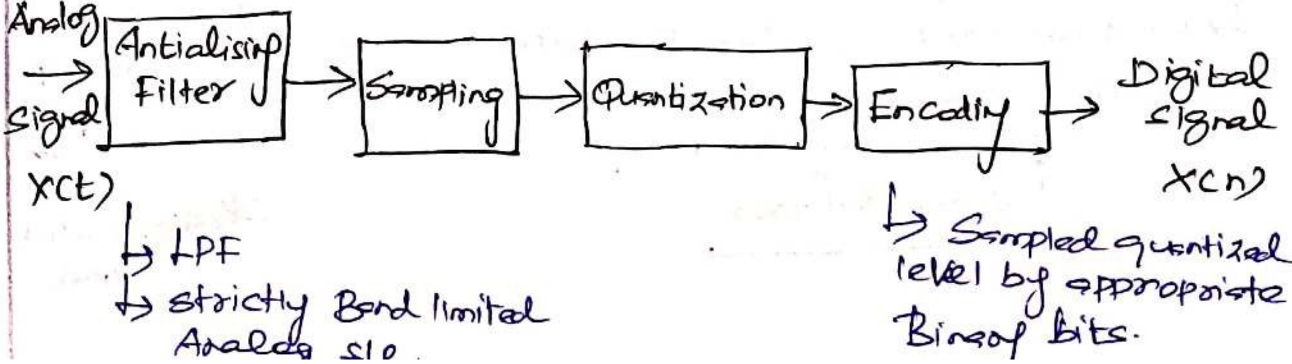
→ The Sampling is the process of Converting Continuous time signal into discrete time signal.

Need for Sampling:

→ Analog signal need to be digitized before being digitally processed.

→ A Continuous time signal can't be processed in the digital processor (or) Computer, to enable the digital transmission of continuous time signal we are using the Sampling.

Digitization of Analog signal.



↳ LPF
↳ strictly Band limited Analog s/g.

↳ Sampled quantized level by appropriate Binary bits.

Quantization:

The process of Converting a discrete time Continuous amplitude signal $x(n)$ into a discrete time discrete amplitude signal $x_q(n)$ is known as Quantization.

Sampling theorem:

A Continuous time signal can be completely represented in its Samples and recovered back if the sampling frequency.

→ If the Sampling frequency is greater than (or) equal to twice the maximum frequency of the signal.

$$\boxed{F_s \geq 2f_{max}}$$

f_s - Sampling frequency
f_{max} - max freq also called 'B' Bandwidth of the signal.

Nyquest Rate:

When the Sampling rate becomes exactly equal to $2f_{max}$ (or) $(2B)$ Samples per second for a given Bandwidth of f_m hertz then its called Nyquest rate

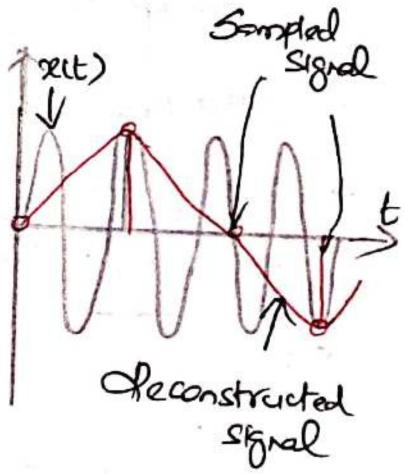
$$F_s = 2f_{max} = 2B$$

$$\text{Nyquest Interval} = \frac{1}{\text{Nyquest rate}} = \frac{1}{2F_{max}} = \frac{1}{2B}$$

Aliasing Effects:

When the high frequency interferes with low frequency and appears as low frequency then the phenomenon is called aliasing.

$$\boxed{F_s < 2f_{max}}$$



Way to Avoid aliasing:

(i) The Sampling Frequency must be higher than 2Fmax

(ii) Use of antialiasing Filter before Sampling.

Quantization Error:

Quantization Error (E) is the difference between original amplitude of the signal at the time of Sampling and its quantized value.

$$\boxed{\text{Quantization Error } (E) = x(nT_s) - x_q(nT_s)}$$

FORMULA:

1. $\sum\limits_{n=0}^{N-1} a^n = \frac{1-a^n}{1-a}, a \neq 1$

2. $\sum\limits_{n=0}^{\infty} a^n = \frac{1}{1-a}$

3. $\sum\limits_{n=0}^{N-1} 1 = N$

4. $\sum\limits_{n=-N}^{N} 1 = 2N+1$

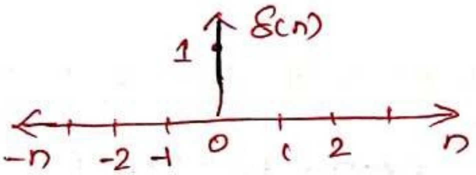
5. $\sum\limits_{n=m}^{\infty} a^n = \frac{a^m}{1-a}, a<1$

6. $\sum\limits_{n=0}^{\infty} 1 = \infty$

7. $\sum\limits_{n=-\infty}^{\infty} 1 = \infty$

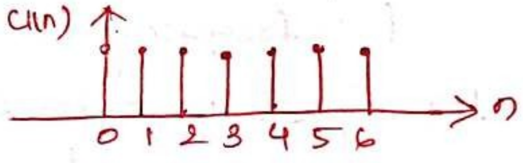
Unit Impulse Sequence

$$\delta(n) = \begin{cases} 1, & n=0 \\ 0, & n \neq 0 \end{cases}$$



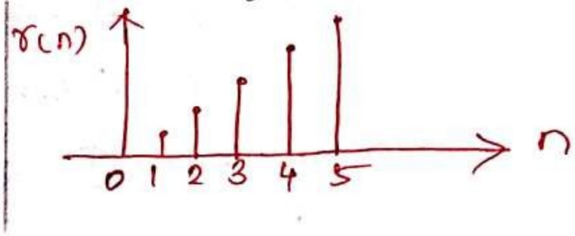
Unit step sequence

$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



Unit Ramp Sequence

$$r(n) = \begin{cases} n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



Problem ①

Consider the analog signal $x(t) = 3\cos 50\pi t + 10\sin 300\pi t - \cos 100\pi t$
What is the nyquist rate for this signal ?

Given: $x(t) = 3\cos 50\pi t + 10\sin 300\pi t - \cos 100\pi t$

$\omega_1 = 50\pi$ $\quad\quad$ $\omega_2 = 300\pi$ $\quad\quad$ $\omega_3 = 100\pi$

$2\pi f_1 = 50\pi$ $\quad\quad$ $2\pi f_2 = 300\pi$ $\quad\quad$ $2\pi f_3 = 100\pi$

$\boxed{f_1 = 25}$ $\quad\quad$ $\boxed{f_2 = 150}$ $\quad\quad$ $\boxed{f_3 = 50}$

To find Nyquist rate $F_s = 2 f_{max}$.

maximum freq $f_{max} = 150$

$F_s = 2 f_{max} = 2 \times 150$

$$\boxed{F_s = 300 \text{ Hz}}$$

Concept of frequency in Discrete Time Signals!

Consider the CT Cosine Wave

$x_a(t) = A\cos(\Omega t + \theta) = A\cos(2\pi F t + \theta)$, $\Omega = 2\pi f$

Equivalent discrete time (DT) cosine wave is represented

$x(n) = A\cos(\omega n + \theta) = A\cos(2\pi f n + \theta)$, $\quad \omega = 2\pi f$ $\rightarrow$ ①

$f \rightarrow$ frequency in cycle per sample.

DT signal is obtained from CT signal by sampling.

$$X(n) = x_a(t)\Big|_{t=nT} = A\cos(2\pi F t + \theta)\Big|_{t=nT}$$

$$= A\cos(2\pi F n T + \theta)$$

$$= A\cos\left(2\pi F n \frac{1}{F_s} + \theta\right) \quad \because F_s = \frac{1}{T}$$

$$= A\cos\left(2\pi \frac{F}{F_s} n + \theta\right) \rightarrow ② \quad T = \frac{1}{F_s}$$

Comparing eqn ① & ②

$$f = \frac{F}{F_s} = FT \text{ (or) } \omega = \frac{\Omega}{F_s} = \Omega T$$

This is the relationship between DT signal frequency and CT signal frequency.

Range of DT signal frequencies:

Sampling theorem $F_s \geq 2 f_{max}$

$$\frac{f_{max}}{F_s} \leq \frac{1}{2} \text{ (or) } f_{max} \leq \frac{1}{2}$$

Generalized notation of writting is,

$$-\frac{1}{2} \leq f_{max} \leq \frac{1}{2}$$

$$-\pi \leq \omega_{max} \leq \pi \quad \text{since } \omega = 2\pi f$$

This is the range of frequencies of DT signal.

DISCRETE FOURIER TRANSFORM (DFT)

DFT Pair: (i) DFT, (ii) IDFT

DFT $$X(k) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi kn}{N}}, \quad k = 0, 1 \cdots N-1$$

IDFT $$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{\frac{j2\pi kn}{N}}, \quad n = 0, 1, 2 \cdots N-1$$

Problem ①

Find the discrete Fourier Transform for $\delta(n)$.

DFT $$X(k) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi kn}{N}}, \quad k = 0, 1, \cdots N-1$$

$$X(k) = \sum_{n=0}^{N-1} \delta(n) e^{\frac{-j2\pi kn}{N}} \qquad \left[\because \delta(n) = \begin{cases} 1, & n=0 \\ 0, & n \neq 0 \end{cases} \right.$$

$$= \delta(0) e^{0}$$

$$\boxed{X(k) = 1}$$

Problem ②

Determine Fourier transform of $x(n) = a^n u(n)$ for $-1 < a < 1$.

Soln: FT formula: $\boxed{X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}}$ Analysis Eqn

$$X(\Omega) = \sum_{n=-\infty}^{\infty} a^n u(n)\, e^{-j\Omega n}$$

$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \sum_{n=0}^{\infty} a^n \cdot 1 \cdot e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} \left(a e^{-j\Omega}\right)^n$$

$$\because \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

$$\boxed{X(\Omega) = \frac{1}{1 - a e^{-j\Omega}}}$$

Inverse DTFT

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega)\, e^{j\Omega n}\, d\Omega$$

Synthesis Equation

Problem ③

Determine the Fourier Transform of $x(n) = u(n)$

Soln: FT formula $\boxed{X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}}$

$$= \sum_{n=-\infty}^{\infty} u(n)\, e^{-j\Omega n}$$

Unit step $u(n) = \begin{cases} 1, & n \geq 0 \\ 0 & \text{o.w} \end{cases}$

$$= \sum_{n=0}^{\infty} 1 \cdot e^{-j\Omega n} \Rightarrow \sum_{n=0}^{\infty} \left(e^{-j\Omega}\right)^n$$

$$\because \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

$$\boxed{X(\Omega) = \frac{1}{1 - e^{-j\Omega}}}$$

4 Point DFT Matrix Formula:

$$W_4 = \begin{array}{c} \\ k=0 \\ k=1 \\ k=2 \\ k=3 \end{array} \begin{array}{cccc} n=0 & n=1 & n=2 & n=3 \\ \hline W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{array}$$

$$[W_4^k] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

IDFT 4 point Matrix Method:

$$[W_4^k]^* = \frac{1}{N} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

Problem ④

Compute the 4-point DFT of the following sequence $x(n) = \{1, 2, 3, 4\}$

Soln:

$$X(k) = x(n) \cdot W_N^k$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1+2+3+4 \\ 1-2j-3+4j \\ 1-2+3-4 \\ 1+2j-3-4j \end{bmatrix} \Rightarrow \begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

$$\boxed{X(k) = \{10, -2+2j, -2, -2-2j\}}$$

Problem ⑤.

Determine the IDFT of 4 point sequence $X(k) = \{10, -2+2j, -2, -2-2j\}$

Soln:

$$x(n) = \frac{1}{N} X(k) (W_N^k)^*$$

$$= \frac{1}{N} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix} \Rightarrow \frac{1}{4} \begin{bmatrix} 10-2-2j-2-2+2j \\ 10+2j-2+2-2j-2 \\ 10+2+2j-2+2-2j \\ 10-2j+2+2+2j+2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 \\ 8 \\ 12 \\ 16 \end{bmatrix}$$

$$\boxed{x(n) = \{1, 2, 3, 4\}}$$

8 point DFT Matrix Method:

$$[W_8^k] = \begin{bmatrix} W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\ W_8^0 & W_8^2 & W_8^4 & W_8^6 & W_8^8 & W_8^{10} & W_8^{12} & W_8^{14} \\ W_8^0 & W_8^3 & W_8^6 & W_8^9 & W_8^{12} & W_8^{15} & W_8^{18} & W_8^{21} \\ W_8^0 & W_8^4 & W_8^8 & W_8^{12} & W_8^{16} & W_8^{20} & W_8^{24} & W_8^{28} \\ W_8^0 & W_8^5 & W_8^{10} & W_8^{15} & W_8^{20} & W_8^{25} & W_8^{30} & W_8^{35} \\ W_8^0 & W_8^6 & W_8^{12} & W_8^{18} & W_8^{24} & W_8^{30} & W_8^{36} & W_8^{42} \\ W_8^0 & W_8^7 & W_8^{14} & W_8^{21} & W_8^{28} & W_8^{35} & W_8^{42} & W_8^{49} \end{bmatrix}$$

$$W_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & (0.707-j0.707) & -j & (-0.707-j0.707) & -1 & (-0.707+j0.707) & j & (0.707+j0.707) \\ 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 & (-0.707-j0.707) & j & (0.707-j0.707) & -1 & (0.707+j0.707) & -j & (-0.707+j0.707) \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & (-0.707+j0.709) & -j & (0.707+j0.707) & -1 & (0.707-j0.707) & j & (-0.707-j0.707) \\ 1 & j & -1 & -j & 1 & j & -1 & -j \\ 1 & (0.707+j0.707) & j & (-0.707+j0.707) & -1 & (-0.707-j0.907) & -j & (0.707-j0.707) \end{bmatrix}$$

Find the Z-Transform of the Sequence $x(n) = \{0.5, 0, 0.5, 0\}$ Using Z-Transform result find its DFT

Soln: Given: $x(n) = \{0.5, 0, 0.5, 0\}$, $N=4$

Z Transform of $x(n)$ is given as,

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}}$$

$$X(z) = \sum_{n=0}^{3} x(n) z^{-n}$$

$$= x(0) z^{0} + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3}$$

$$= 0.5 + 0 z^{-1} + 0.5 z^{-2} + 0 z^{-3}$$

$$\boxed{X(z) = 0.5 + 0.5 z^{-2}}$$

The z-Transform and DFT are related as,

$$\boxed{X(K) = X(z)\Big|_{z = e^{j\frac{2\pi k}{N}}}}$$

$$X(K) = \left(0.5 + 0.5 z^{-2}\right)\Big|_{z = e^{j\frac{2\pi k}{N}}}$$

$$= 0.5 + 0.5\left(e^{\frac{j2\pi k}{\cancel{4}}}\right)^{-2} \Rightarrow 0.5 + 0.5 e^{-\frac{\cancel{4}\pi k j}{\cancel{4}}}$$

$$= 0.5 + 0.5 e^{-j\pi K}$$

$$= 0.5 + 0.5(\cos\pi - j\sin\pi)^{k}$$

$[e^{-j\theta} = \cos\theta - j\sin\theta$

$$\boxed{X(K) = 0.5 + 0.5(-1)^{k}}$$

Put $k = 0, 1, 2, 3$

$k=0$; $0.5 + 0.5(-1)^{0} \Rightarrow 1$

$k=1$; $0.5 + 0.5(-1)^{1} \Rightarrow 0$

$k=2$; $0.5 + 0.5(-1)^{2} \Rightarrow 1$

$k=3$; $0.5 + 0.5(-1)^{3} \Rightarrow 0$

$$\boxed{X(K) = \{1, 0, 1, 0\}}$$

(iii) $x(n) = \delta(n-n_0)$

Consider a circular time shift property:

$$\boxed{x((n-m))_N \overset{DFT}{\underset{N}{\longleftrightarrow}} X(K)\, e^{-\frac{j2\pi K m}{N}}}$$

$x(n) = \delta(n)$. Hence $X(K) = 1$. Let $m = n_0$ in above eqn.

$$\delta(n-n_0)_N \overset{DFT}{\underset{N}{\longleftrightarrow}} 1 \cdot e^{\frac{-j2\pi K n_0}{N}}$$

$$\boxed{\delta(n-n_0)_N \overset{DFT}{\underset{N}{\longleftrightarrow}} e^{\frac{-j2\pi K n_0}{N}}}$$

(iv) $x(n) = e^{-0.5n}, \quad 0 \le n \le 5$

Consider, $x(n) = a^n$ for $0 \le n \le N-1$

its given that $x(n) = e^{-0.5n}, \quad 0 \le n \le 5$, hence we get

$$a^n = \left(e^{-0.5}\right)^n, \quad N = 6$$

The DFT of a^n for $0 \le n \le N-1$ given as

$$\boxed{X(K) = \frac{1 - a^N}{1 - a e^{-j2\pi K/N}}}$$

Puting for $a = e^{-0.5}$ and $N = 6$

$$X(K) = \frac{1 - \left(e^{-0.5}\right)^6}{1 - e^{-0.5}\, e^{-\frac{j\cancel{2}\pi K}{\cancel{6}}}} \Rightarrow \frac{1 - e^{-0.5 \times 6}}{1 - e^{-0.5}\, e^{-\frac{j\pi K}{3}}}$$

$$\boxed{X(K) = \frac{0.95}{1 - \left(0.6 \cdot e^{\frac{-j\pi K}{3}}\right)}}$$

Compute the number of multiplication needed in the FFT Computation of DFT of a 32 point sequence.

<u>Soln:</u> Number multiplication Needed in FFT Computation of 32 point $= \frac{N}{2} \cdot \log_2 N$; $N=32$

$$= \frac{32}{2} \log_2 32$$

$$= 16 \log_2 32$$

$$= 16 \times 5$$

$$= \boxed{80 \text{ No.of multiplication needed.}}$$

$\log_2 32 \Rightarrow \frac{\log 32}{\log 2}$

$= 5$

The number of Points is given by $N=64$ Compute the number of Complex multiplication and Addition required to Perform DFT and FFT.

<u>Soln:</u> $N=64$

DFT

① No.of Addition : $N(N-1)$

$= 64(64-1)$

$= \boxed{4032}$

② No.of Multiplication : N^2

$= 64^2$

$= \boxed{4,096}$

FFT

① No.of Addition : $N \log_2 N$

$= 64 \log_2 46$

$= 64 \frac{\log 46}{\log 2}$

$= \boxed{384}$

② No of Multiplication :

$\frac{N}{2} \log_2 N \Rightarrow \frac{64}{2} \log_2 64$

$\Rightarrow 32 \frac{\log 64}{\log 2} \Rightarrow \boxed{192}$

What is the relation between Z-transform and DFT?

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$\boxed{X(k) = X(z) \Big|_{z = e^{\frac{-j2\pi k}{N}}}}$$

Properties of DFT:

(i) Periodicity
(ii) Time shifting
(iii) Time Reversal
(iv) Conjucate symmetry
(V) Linearity
(Vi) Multiplication of two DFT and convolution
(vii) Parsevals theorem
(viii) Circular Convolution.

Periodicity:

Let $x(n)$ and $X(k)$ be the DFT pair then

$$\boxed{\begin{aligned} x(n+N) &= x(n) \quad \text{for all } n \\ X(k+N) &= X(k) \quad \text{For all } k. \end{aligned}}$$

Proof: DFT formula $X(k) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi kn}{N}}$

$$X(k+N) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi(k+N)n}{N}}$$

$$X(k+N) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi kn}{N}} \cdot e^{\frac{-j2\pi \cancel{N}n}{\cancel{N}}}$$

$e^{-j2\pi n} = 1$

$\cos 2\pi n - j\sin 2\pi n = 1$

$$X(k+N) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi kn}{N}} \cdot 1$$

$$\boxed{X(k+N) = X(k)}$$

∴ Hence Prooved.

Time Reversal:

$$\boxed{\begin{aligned} &\text{if } x(n) \xleftrightarrow[N]{DFT} X(k) \text{ then} \\ &x[N-n] \xleftrightarrow[N]{DFT} X(N-k) \end{aligned}}$$

Proof: $DFT[x(n)] = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi kn}{N}}$

$$DFT\left[x(N-n)\right] = \sum_{n=0}^{N-1} x(N-n)\, e^{\frac{-j2\pi kn}{N}}$$

Put $m = N-n \Rightarrow n = N-m$

Because of circular time shift Limit 'm' various from 0 to N-1

$$= \sum_{m=0}^{N-1} x(m)\, e^{\frac{-j2\pi k\,(N-m)}{N}}$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{-j2\pi k}\, e^{\frac{j2\pi km}{N}}$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{\frac{j2\pi km}{N}} \qquad \left[\because e^{-j2\pi k} = 1\right.$$

Multiply by $e^{\frac{-j2\pi mN}{N}}$

$$= \sum_{m=0}^{N-1} x(m)\, e^{\frac{j2\pi km}{N}}\, e^{\frac{-j2\pi mN}{N}}$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{\frac{-j2\pi m(N-k)}{N}}$$

$$\boxed{DFT\left[x(N-n)\right] = X(N-k)}$$

$\therefore$ Hence proved.

Circular time shift:

$$\boxed{\begin{array}{l} \text{If } x(n) \xleftrightarrow[N]{DFT} X(k) \text{ then} \\ x(n-m) \xleftrightarrow[N]{DFT} X(k)\, e^{\frac{-j2\pi km}{N}} \end{array}}$$

Proof: $DFT\left[x(n)\right] = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}}$

$$DFT\left[x(n-m)\right] = \sum_{n=0}^{N-1} x(n-m)\, e^{\frac{-j2\pi kn}{N}}$$

Put $p = n - m \Rightarrow n = p + m$

$$= \sum_{p=0}^{N-1} x(p)\, e^{\frac{-j2\pi k(p+m)}{N}}$$

$$= \sum_{p=0}^{N-1} x(p)\, e^{\frac{-j2\pi kp}{N}}\, e^{\frac{-j2\pi km}{N}}$$

$$= e^{\frac{-j2\pi km}{N}} \sum_{p=0}^{N-1} x(p)\, e^{\frac{-j2\pi kp}{N}}$$

$$DFT\left[x(n-m)\right] = e^{\frac{-j2\pi km}{N}}\, X(k)$$

$$\boxed{DFT\left[x(n-m)\right] = X(k)\, e^{\frac{-j2\pi km}{N}}}$$

∴ Hence proved.

Conjucate symmetry:

$$\boxed{\begin{array}{l} \text{If } x(n) \xleftrightarrow[N]{DFT} X(k) \text{ then} \\ x^*(n) \xleftrightarrow[N]{DFT} X^*(N-k) \end{array}}$$

Proof: $DFT\left[x(n)\right] = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}}$

$$DFT\left[x^*(n)\right] = \sum_{n=0}^{N-1} x^*(n)\, e^{\frac{-j2\pi kn}{N}}$$

$$= \left[\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}}\right]^*$$

Multiply by $e^{\frac{-j2\pi nN}{N}}$

$$= \left[\sum_{n=0}^{N-1} x(n)\, e^{\frac{j2\pi kn}{N}} \cdot e^{\frac{-j2\pi nN}{N}}\right]^*$$

$$= \left[\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi(N-k)n}{N}}\right]^*$$

$\Rightarrow \left(X(N-k)\right)^*$

$$\boxed{DFT\left[x^*(n)\right] = X^*(N-k)}$$

Hence proved.

Circular Convolution:

$$\boxed{\begin{array}{l} \text{if } x(n) \xleftrightarrow[N]{DFT} X(k) \text{ then} \\ x_1(n) \circledast x_2(n) \xleftrightarrow[N]{DFT} X_1(k)\cdot X_2(k) \end{array}}$$

Proof: DFT formula $X(k) = \sum_{n=0}^{N-1} x_1(n)\, e^{\frac{-j2\pi kn}{N}}$

Let $n = m$, $X_1(k) = \sum_{m=0}^{N-1} x_1(m)\, e^{\frac{-j2\pi km}{N}} \rightarrow ①$

and $\underline{\underline{n=p}}$ $X_2(k) = \sum_{p=0}^{N-1} x_2(p)\, e^{\frac{-j2\pi kp}{N}} \rightarrow ②$

W.K.T

IDFT: $IDFT[X_1(m)\, X_2(n)] = \frac{1}{N}\sum_{k=0}^{N-1} X_1(k)\, X_2(k)\, e^{\frac{j2\pi kn}{N}} \rightarrow ③$

Sub $X_1(k)$, $X_2(k)$ in ③

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{m=0}^{N-1} x_1(m)\, e^{\frac{-j2\pi km}{N}} \sum_{p=0}^{N-1} x_2(p)\, e^{\frac{-j2\pi kp}{N}}\right]\cdot e^{\frac{j2\pi kn}{N}}$$

Let $p = n-m$; $\frac{1}{N}\sum_{m=0}^{N-1} x_1(m) \sum_{p=0}^{N-1} x_2(p) \sum_{k=0}^{N-1} e^{\frac{-j2\pi k(m+p-n)}{N}}$

$p = n-m$

$$\frac{1}{N}\sum_{m=0}^{N-1} x_1(m) \sum_{m=0}^{N-1} x_2(n-m) \sum_{k=0}^{N-1} 1$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} x_1(m) \sum_{m=0}^{N-1} x_2(n-m)\cdot N \qquad \left[\because \sum_{k=0}^{N-1} 1 = N\right.$$

$$= \sum_{m=0}^{N-1} x_1(m)\, x_2(n-m)$$

$$= x_1(n) * x_2(n)$$ [∵ By definition of circular convolution]

hence, $IDFT[X_1(k) X_2(k)] = x_1(n) \circledast x_2(n)$

that $\boxed{DFT[x_1(n) * x_2(n)] = X_1(k)\, X_2(k)}$

∴ Hence proved.

Linearity property:

Let $x_1(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X_1(k)$ and $x_2(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X_2(k)$, then

$$a_1x_1(n) + a_2x_2(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} a_1X_1(k) + a_2X_2(k)$$

Proof:

$$DFT\left[x(n)\right] = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi kn}{N}}$$

$$= \sum_{n=0}^{N-1} \left[a_1x_1(n) + a_2x_2(n)\right] e^{\frac{-j2\pi kn}{N}}$$

$$= \sum_{n=0}^{N-1} a_1x_1(n) e^{\frac{-j2\pi kn}{N}} + \sum_{n=0}^{N-1} a_2x_2(n) e^{\frac{-j2\pi kn}{N}}$$

$$= a_1 \sum_{n=0}^{N-1} x_1(n) e^{\frac{-j2\pi kn}{N}} + a_2 \sum_{n=0}^{N-1} x_2(n) e^{\frac{-j2\pi kn}{N}}$$

$$\boxed{DFT\left[x(n)\right] = a_1 X_1(k) + a_2 X_2(k)}$$

∴ Hence proved.

Multiplication of Two sequence:

if $x_1(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X_1(k)$ and $x_2(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X_2(k)$

$$x_1(n)\, x_2(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} \frac{1}{N} X_1(k) \; N \; X_2(k)$$

Parseval's Theorem:

If $x(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X(k)$ and $y(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} Y(k)$

$$\sum_{n=0}^{N-1} x(n)\, y^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, Y^*(k)$$

Find DFT of 4 point sequence $x(n) = \cos\left(\frac{n\pi}{4}\right)$

Soln: Given: $x(n) = \cos\left(\frac{n\pi}{4}\right)$, $N = 4$.

Find the input sequence, put $n = 0, 1, 2, 3$.

$n=0$: $x(0) = \cos\left(\frac{0 \times \pi}{4}\right) = \cos 0 \Rightarrow \boxed{x(0) = 1}$

$n=1$: $x(1) = \cos\left(\frac{1 \times \pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) \Rightarrow \boxed{x(1) = 0.7071}$

$n=2$: $x(2) = \cos\left(\frac{2\pi}{4}\right) = \cos(\pi/2) = \boxed{x(2) = 0}$

$n=3$: $x(3) = \cos\left(\frac{3\pi}{4}\right) = \boxed{x(3) = -0.7071}$

$$X_4 = [W_4]\, x_4$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 0.7071 \\ 0 \\ -0.7071 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + \cancel{0.7071} + 0 - \cancel{0.7071} \\ 1 - 0.7071j - 0 - 0.7071j \\ 1 - \cancel{0.7071} + 0 + \cancel{0.7071} \\ 1 + 0.7071j - 0 + 0.7071j \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 - j1.4142 \\ 1 \\ 1 + j1.4142 \end{bmatrix}$$

$$\boxed{[X(k)]_4 = \{1,\ 1 - j1.4142,\ 1,\ 1 + j1.4142\}}$$

Problem:-

Evaluate the 8-point DFT for the following sequence using DIT-FFT Algorithm

$$x(n) = \begin{cases} 1 & \text{for } -3 \le n \le 3 \\ 0 & \text{for otherwise} \end{cases}$$

Given: $x(n) = \begin{cases} 1 & \text{For } -3 \le n \le 3 \\ 0 & \text{For otherwise} \end{cases}$

$x(n) = \{1, 1, 1, \underset{\underset{x(0)}{\uparrow}}{1}, 1, 1, 1\}$ Here $N = 7$

We require 8 point DFT let us append one zero at the End above sequence to Make $N = 8$ Samples.

$x(n) = \{1, 1, 1, \underset{\underset{x(0)}{\uparrow}}{1}, 1, 1, 1, 0\} \Rightarrow N = 8$

$x(n) = \{\underset{\underset{x(0)}{\downarrow}}{1}, 1, 1, 1, 0, 1, 1, \underset{\underset{x(7)}{\downarrow}}{1}\}$ For $n = 0$ to 7

$x(0) \ldots\ldots\ldots\ldots x(7)$

To find Twiddle Factor $W_N^n = e^{\frac{-j2\pi n}{N}}$, $N = 8$, $n = 0, 1, 2, 3.$

$n=0$: $W_8^0 = e^{-j\frac{2\pi\times 0}{8}} = e^0 = \boxed{1 \Rightarrow W_8^0}$

$n=1$: $W_8^1 = e^{-j\frac{2\pi\times 1}{8}} = e^{-j\pi/4} \Rightarrow \cos\pi/4 - j\sin\pi/4$

$$\boxed{W_8^1 = 0.707 - j0.707}$$

$n=2$: $W_8^2 = e^{-j\frac{2\pi\times 2}{8}} \Rightarrow e^{-j\pi/2} \Rightarrow \cos\pi/2 - j\sin\pi/2$

$$\boxed{W_8^2 = -j}$$

$n=3$: $W_8^3 = e^{-j\frac{2\pi\times 3}{8}} = e^{-j\frac{3\pi}{4}} \Rightarrow \cos\pi/4 - j\sin\pi/4$

$$\boxed{W_8^8 = -0.707 - j0.707}$$

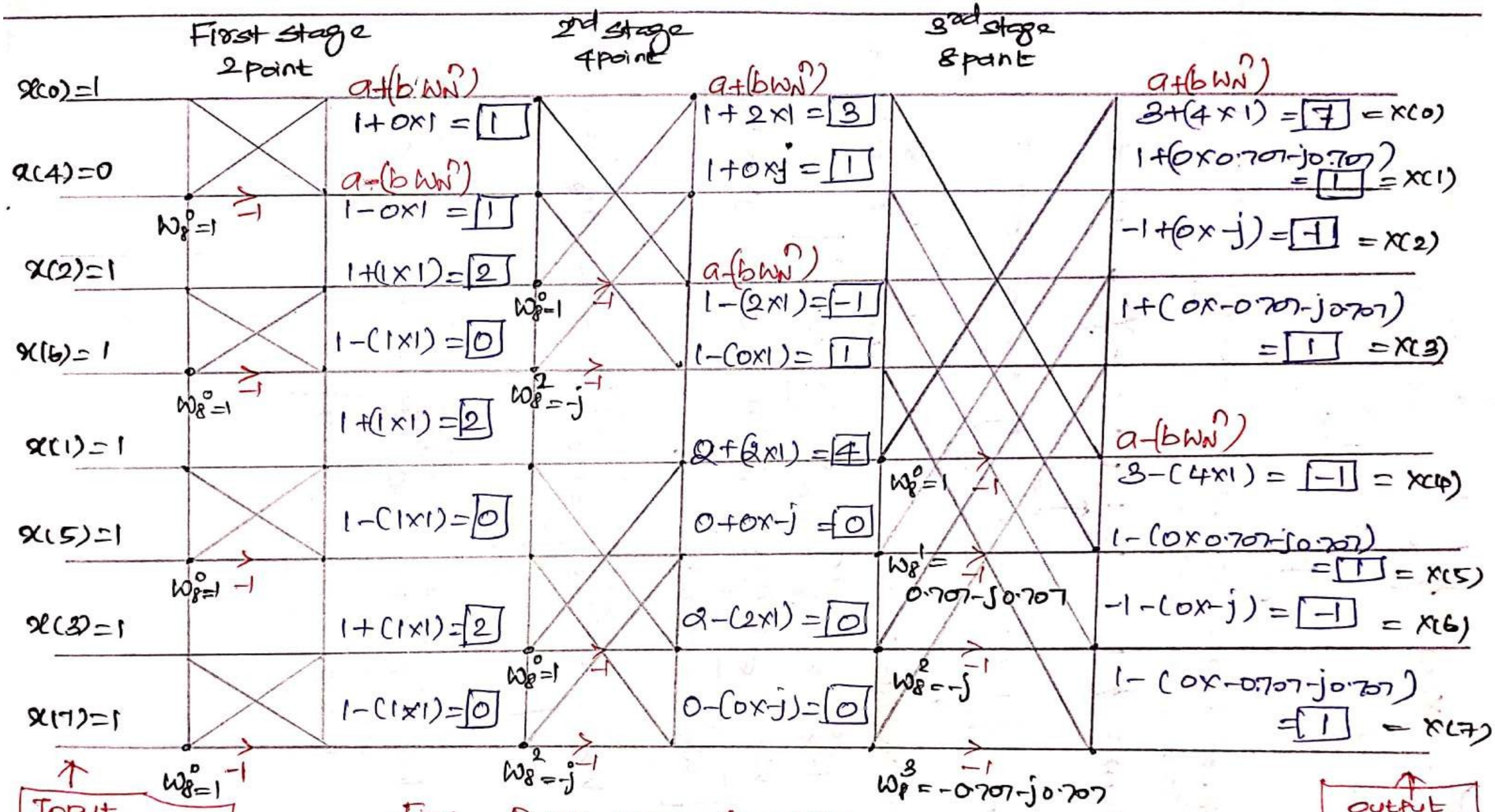


Fig: DIT-FFT Algorithm.

$$\text{Ans: } \boxed{X(k)=\{7, 1, -1, 1, -1, 1, -1, 1\}}$$

Find the 8-point DFT of $\{2,1,2,1\}$ using DIF-FFT
Draw the signal flow graph for $N=8$ with intermediate value, stuff appropriate zeros

Soln: Given: $x(n) = \{2, 1, 2, 1\}$, $N = 8$

To find 8-point DFT: by adding 4 zeros to the sequence Now $x(n) = \{\underset{x(0)}{\underset{\downarrow}{2}}, 1, 2, 1, 0, 0, 0, \underset{x(7)}{\underset{\downarrow}{0}}\}$

To find Twiddle Factor

$$\boxed{W_N^n = e^{\frac{-j2\pi n}{N}}} \quad . \quad N=8, \; n=0,1,2,3$$

n=0, $W_8^0 = e^{\frac{-j2\pi\times0}{8}} = e^0 \Rightarrow \boxed{W_8^0 = 1}$

n=1: $W_8^1 = e^{\frac{-j2\pi\times1}{8}} = e^{-j\pi/4} = \cos\pi/4 - j\sin\pi/4$

$$\boxed{W_8^1 = 0.707 - j0.707}$$

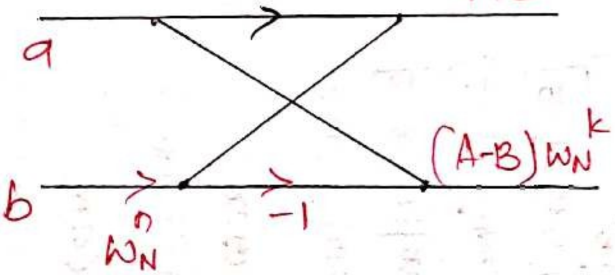
n=2: $W_8^2 = e^{\frac{-j2\pi\times2}{8}} \Rightarrow e^{-j\pi/2} \Rightarrow \cos\pi/2 - j\sin\pi/2$

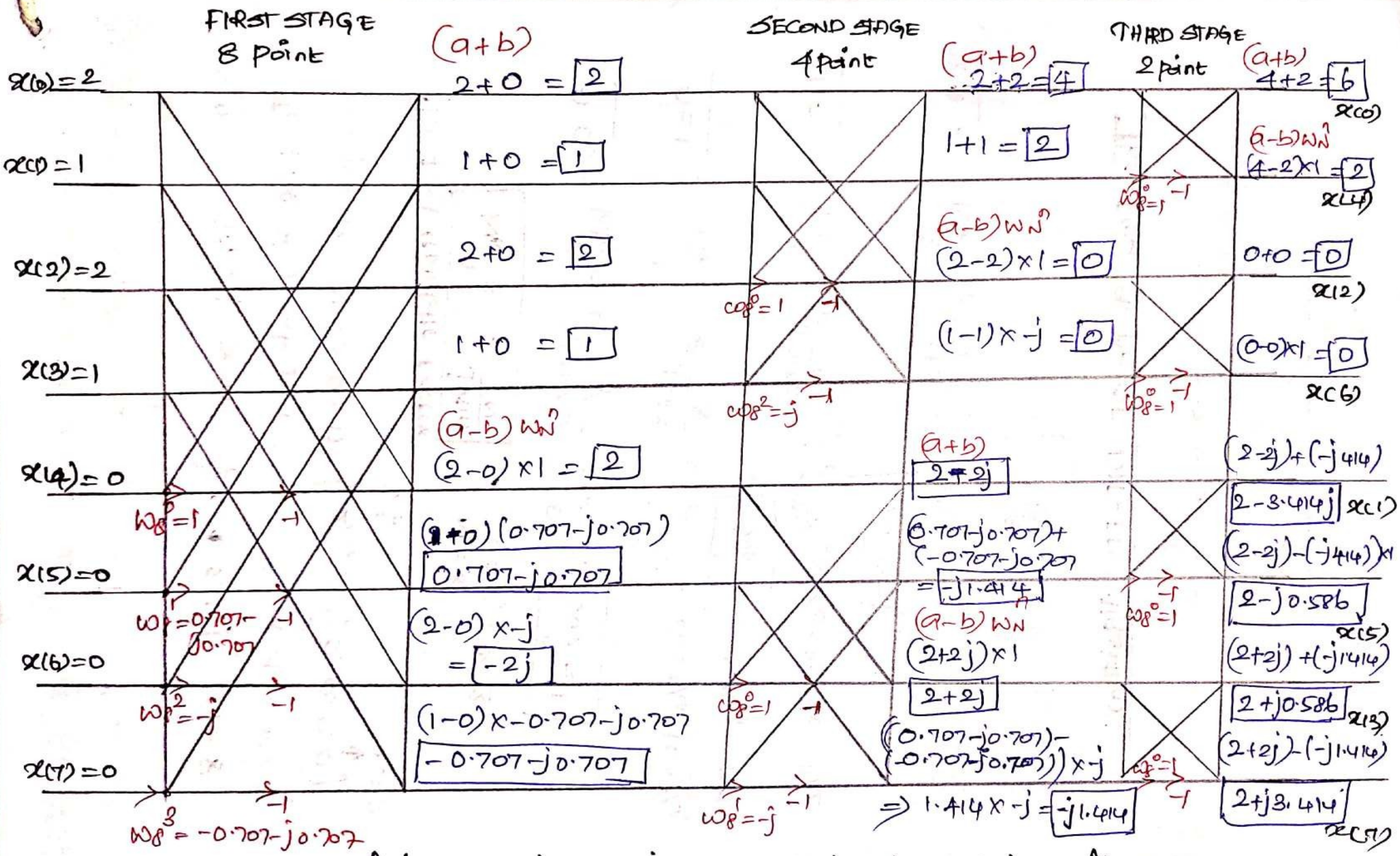
$$\boxed{W_8^2 = -j}$$

n=8: $W_8^3 = e^{\frac{-j2\pi\times3}{8}} = e^{-j3\pi/4} \Rightarrow \cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4}$

$$\boxed{W_8^3 = -0.707 - j0.707}$$

The Flow graph of DIF-FFT Algorithm:





Ans: $X(k) = \{ 6, 2-3.414j, 0, 2+j0.586, 2, 2-j0.586, 0, 2+j3.414 \}$

Find the IDFT of the sequence $X(k) = \{10, -2+2j, -2, -2-2j\}$ using DIT algorithm.

Soln: Given: $X(k) = \{10, -2+2j, -2, -2-2j\}$

$\boxed{N=4.}$

Take Conjicate of X(k)

$$X^{*}(k) = \{\underset{x(0)}{10}, \underset{x(1)}{-2-2j}, \underset{x(2)}{-2}, \underset{x(3)}{-2+2j}\}$$

Find Twiddle Factor!

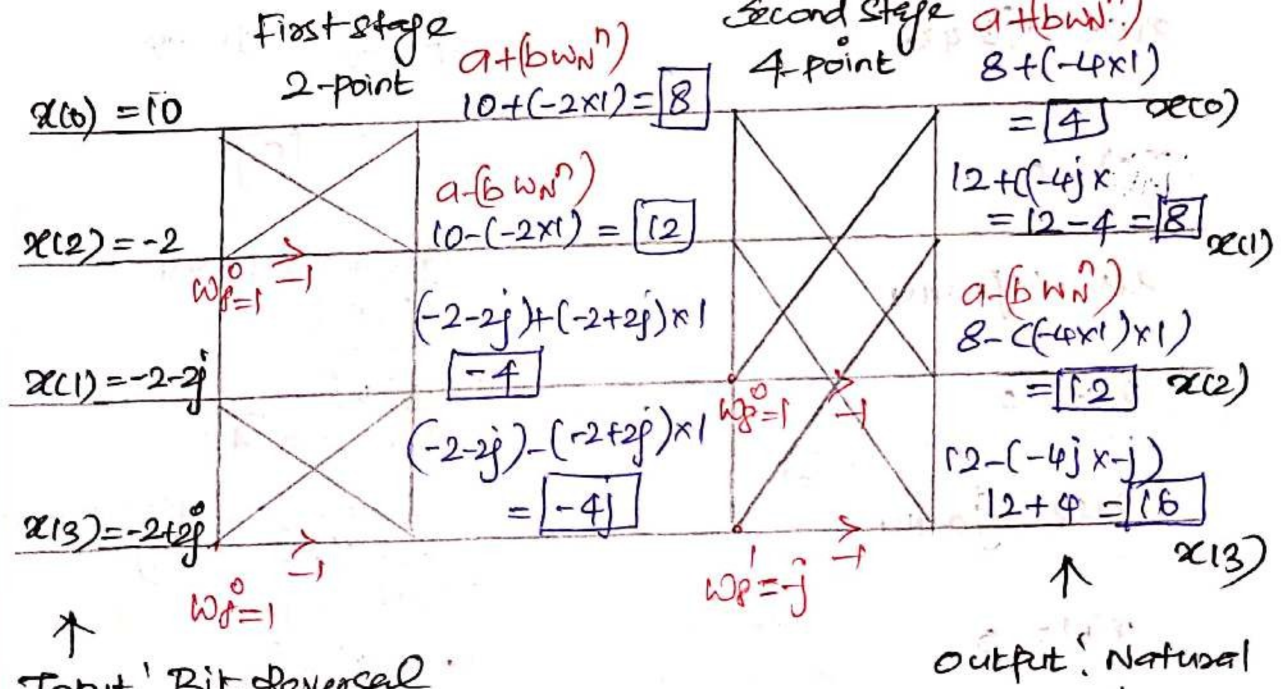
$$W_N^n = e^{-j\frac{2\pi n}{N}}, \quad n = 0, 1,$$

$$n=0; \quad e^{-j\frac{2\pi \times 0}{4}} = e^{0} = \boxed{W_4^0 = 1}$$

$$W_4^{*} = e^{-j\frac{2\pi \times 1}{4}} = e^{-j\pi/2} \Rightarrow \cos\pi/2 - j\sin\pi/2$$

$$\boxed{W_4^1 = -j}$$

DIT-FFT Algorithm:



$$\boxed{X(n) = \{4, 8, 12, 16\}}$$

Find the IDFT of the sequence $X(K) = \{4, 1-j2.414, 0, 1-j0.414, 0, 1+j0.414, 0, 1+j2.414\}$ Using DIF algorithm.

Soln: To take conjicate $X^*(K)$

$= \{\underset{\underset{x(0)}{\uparrow}}{4}, 1+j2.414, 0, 1+j0.414, 0, 1-j0.414, 0, \underset{\underset{x(7)}{\downarrow}}{1j2.414}\}$

To Find Twiddle factor: $\boxed{W_N^n = e^{\frac{-j2\pi n}{N}}}$

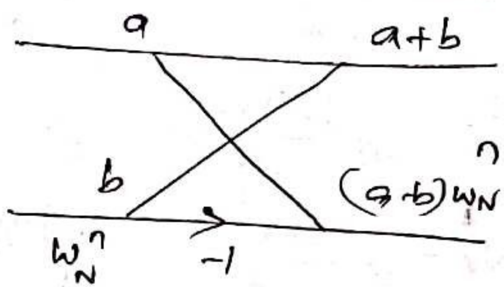
$w_8^0 = 1$

$w_8^1 = 0.707-j0.707$

$w_8^2 = -j$

$w_8^3 = -0.707-j0.707$

Basic structure:



DIF Algorithm:

1st Stage
8 point

$x(0) = 4$

$x(1) = 1+j2.414$

$x(2) = 0$

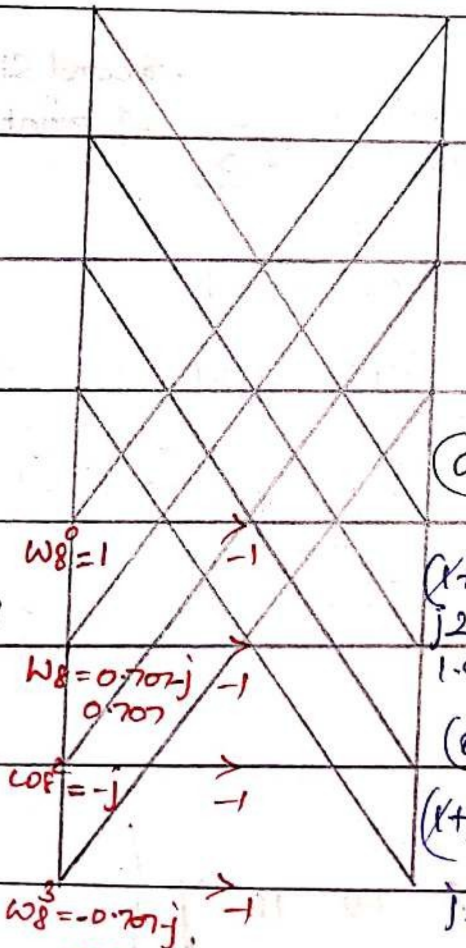
$x(3) = 1+j0.414$

$x(4) = 0$

$x(5) = 1-j0.414$

$x(6) = 0$

$x(7) = 1-j2.414$



$(a+b)$

$4+0 = \boxed{4}$

$1+j2.414+1-j0.414$

$\boxed{2+2j}$

$0+0 = \boxed{0}$

$1+j0.414+1-j2.414$

$\boxed{2-2j}$

$(a-b)W_N^n$

$(4-0)\times 1 = \boxed{4}$

$(\cancel{1}+j2.414-\cancel{1}+j0.414)(0.707-j0.707)$

$j2.828\times(0.707-j0.707)$

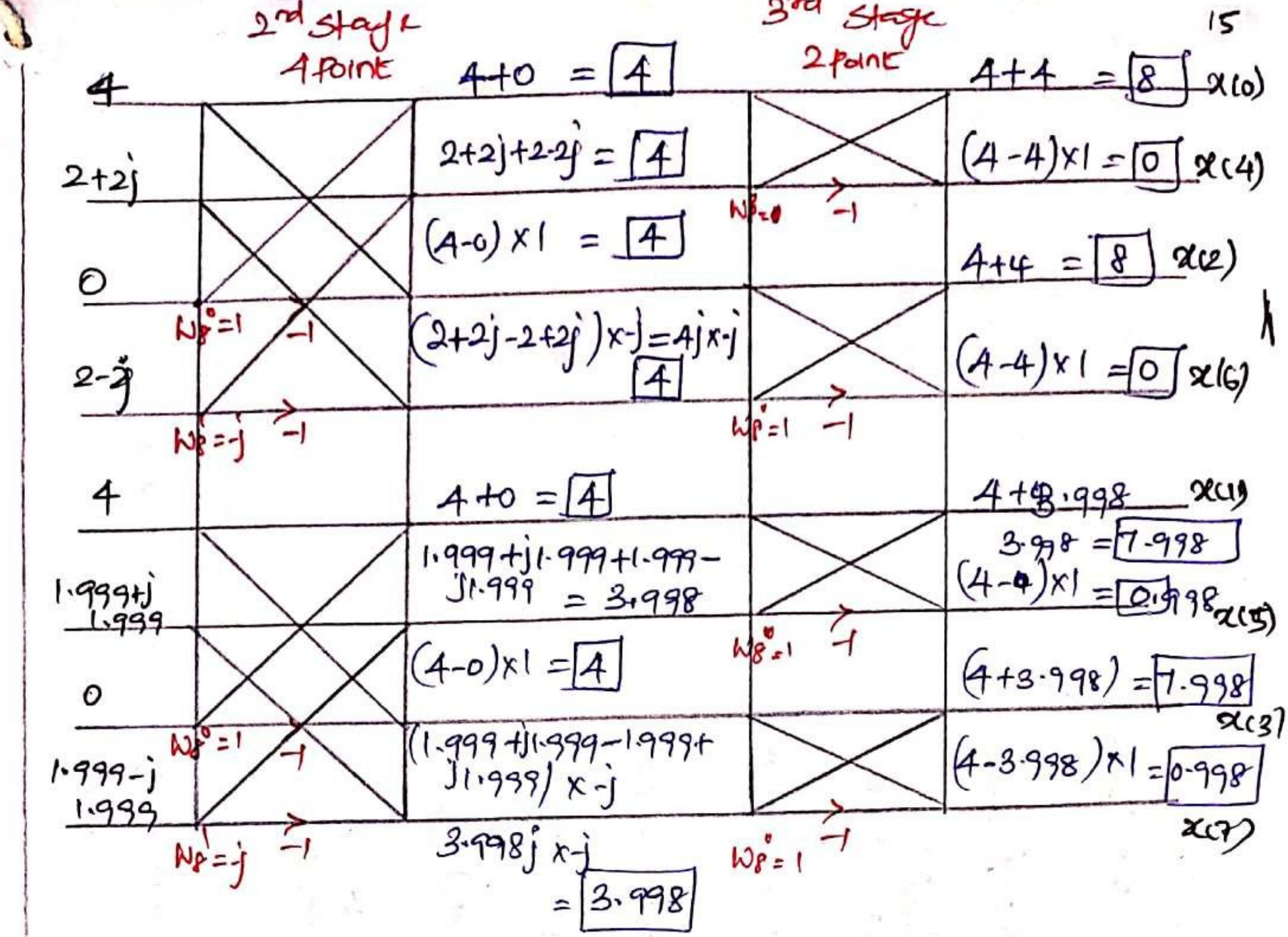
$1.999+1.999j$

$(0-0)\times j = \boxed{0}$

$((\cancel{1}+j0.414-\cancel{1}+j2.414)\times -0.707-j0.707$

$j2.828\times(-0.707-j0.707)$

$-1.999j+1.999$



$$X(n) = \frac{1}{N}\{8, 0, 8, 0, 7.998, 0.998, 7.998, 0.998\}$$

$$X(n) = \frac{1}{8}\{8, 0, 8, 0, 8, 1, 8, \}$$

$$\boxed{X(n) = \{1, 0, 1, 0, 1, 1/8, 1, 1/8\}}$$

Compute the FFT using DIT algorithm for the sequence $x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$ and draw the corresponding flow diagram

Given: $N = 8$, $x(n) = \{\underset{\substack{\uparrow \\ x(0)}}{1}, 2, 3, 4, 4, 3, 2, \underset{\substack{\uparrow \\ x(7)}}{1}\}$

Twiddle factor $\boxed{W_N^n = e^{-j2\pi n/N}}$, $n = 0, 1, 2, 3$, $N = 8$

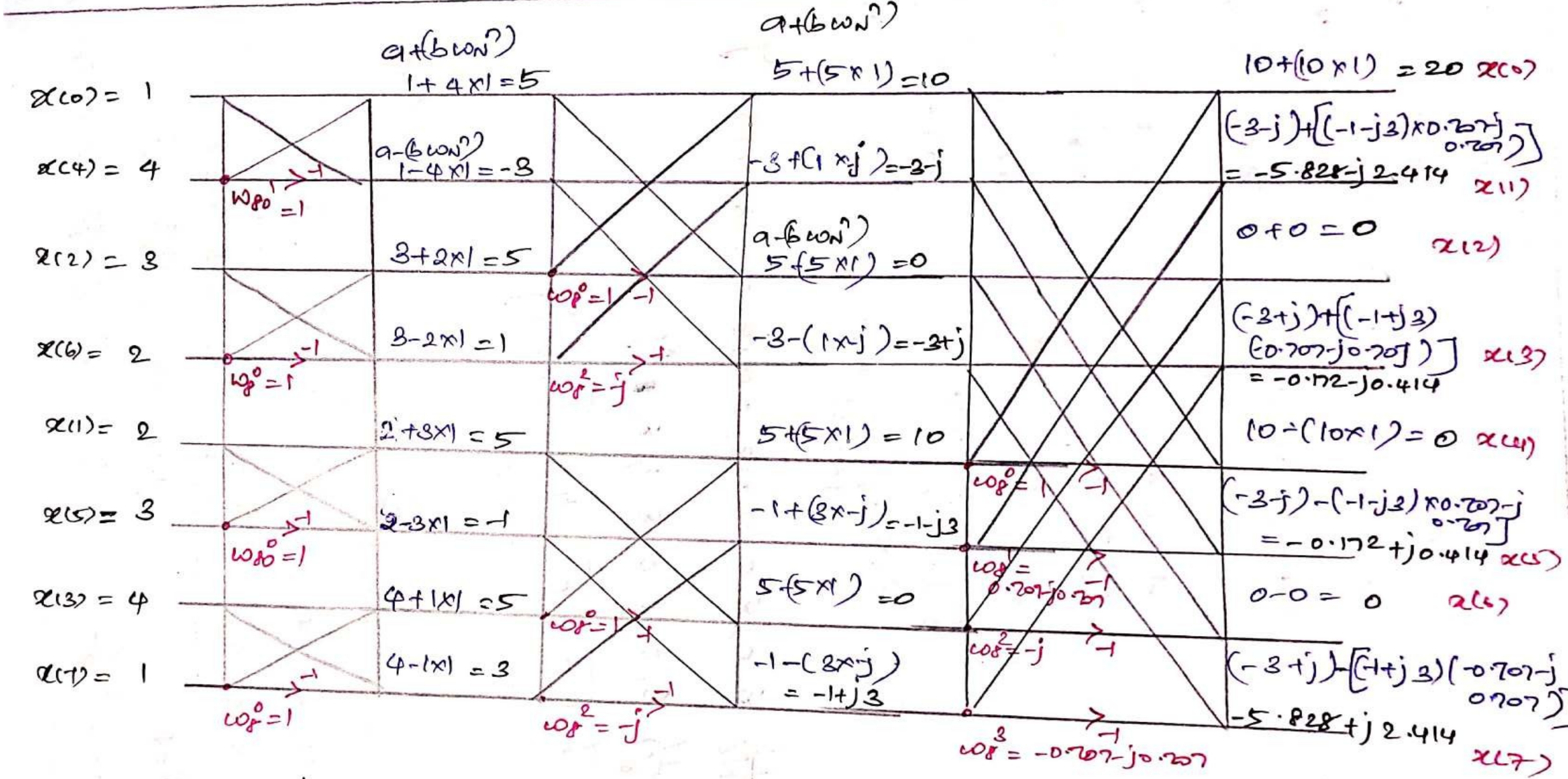
$n=0$, $W_8^0 = e^0 = 1$

$n=2$, $W_8^2 = e^{-j\frac{2\pi \times 2}{8}} = -j$

$n=1$, $W_8^1 = e^{-j\frac{2\pi \times 1}{8}}$

$= 0.707 - j0.707$

$n=3$, $W_8^3 = e^{-j\frac{2\pi \times 3}{8}} = -0.707 - j0.707$



$X(k) = \{ 20, -5.828-j2.414, 0, -0.172-j0.414, 0, -0.172+j0.414, 0, -5.828+j2.414 \}$

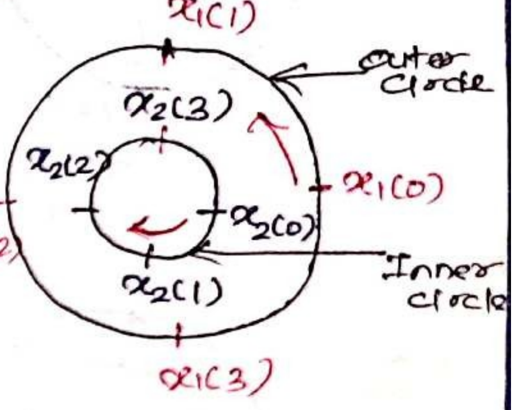
CIRCULAR CONVOLUTION:

CONCENTRIC CIRCULAR METHOD:

The circular convolution of two sequences $x_1(n)$ & $x_2(n)$ is defined as,

$$\boxed{\begin{aligned} y(n) &= x_1(n) \, Ⓝ \, x_2(n) \\ y(n) &= \sum_{m=0}^{N-1} x_1(m) \, x_2[(n-m)]_N \end{aligned}}$$

Steps in Concentric Circular Method:



Step 1: Plot 'N' samples of $x_1(n)$ as equally spaced points around an outer circle in counter clockwise direction.

Step 2: Start at the same point as $x_1(n)$, plot the $x_2(n)$ around an inner circle in clockwise direction.

Step 3: Multiply the corresponding samples on two circles and sum the products to produce the output

$$\boxed{y(0) = x_1(0) \, x_2(0) + x_1(1) \, x_2(3) + x_1(2) \, x_2(2) + x_2(1) \, x_1(3)}$$

Step 4: Rotate the inner circle one sample at a time in counter clockwise & repeat step ③.

Step 5: Repeat the step ④ until the first sample of $x_1(n)$ reaches the first sample of $x_2(n)$.

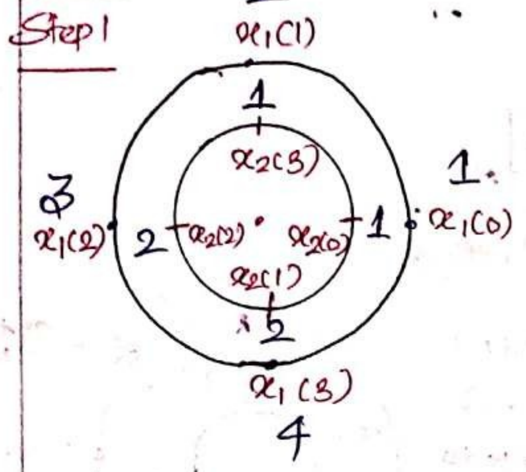
Problem ①:

Find the circular Convolution of given sequences.

$x_1(n) = \{1, 2, 3, 4\}$, $x_2(n) = \{1, 2, 2, 1\}$.

Soln:

Step 1



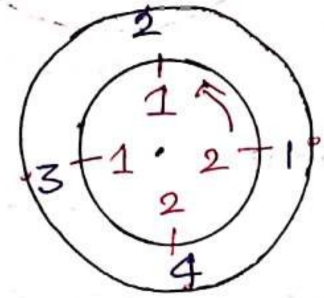
$$x_1(n) = \{\underset{x_1(0)}{\underset{\downarrow}{1}}, \underset{x_1(1)}{\underset{\downarrow}{2}}, \underset{x_1(2)}{\underset{\downarrow}{3}}, \underset{x_1(3)}{\underset{\downarrow}{4}}\}.$$

$$x_2(n) = \{\underset{x_2(0)}{\underset{\downarrow}{1}}, \underset{x_2(1)}{\underset{\downarrow}{2}}, \underset{x_2(2)}{\underset{\downarrow}{2}}, \underset{x_2(3)}{\underset{\downarrow}{1}}\}$$

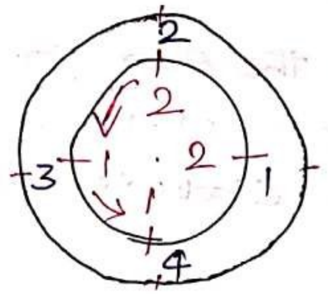
$y(0) = 1+2+6+8$

$\boxed{y(0) = 17}$

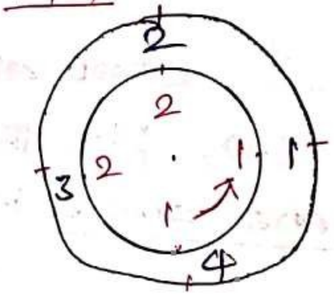
Step 2



Step 3



Step 4



$y(1) = 2\times1 + 2\times1 + 3\times1 + 2\times4$

$\boxed{y(1) = 15}$

$y(2) = 2+4+3+4$

$\boxed{y(2) = 13}$

$y(3) = 1+4+6+4$

$\boxed{y(3) = 15}$

Ans: $\boxed{y(n) = \{17, 15, 13, 15\}}$

HOW TO OBTAIN LINEAR CONVOLUTION FROM CIRCULAR CONVOLUTION ?

Consider the length of $x_1(n) = L_1$, $x_2(n) = L_2$

In Linear Convolution, the length of the output sequence is $\boxed{L_1 + L_2 - 1} = N$

Eg:

$L_1 = 3,\ L_2 = 2$

$N = L_1 + L_2 - 1 \Rightarrow 3+2-1$ $\boxed{N=4}$

→ Circular Convolution, the length of output Sequence is $\boxed{N = \max(L_1, L_2)}$

Eg: $L_1 = 3,\ L_2 = 2$

$N = \max(3, 2) \Rightarrow \boxed{N = 3}$

→ The Linear Convolution can be obtained by increasing the lengths of the sequences $x_1(n)$ & $x_2(n)$ to $L_1 + L_2 - 1$ samples.

→ In order to obtain the number of samples in circular convolution equal to L_1+L_2-1, Both $x_1(n)$, $x_2(n)$ must be appended with appropriate number of zero valued samples.

Relation between Linear Convolution & Circular Convolution

Linear Convolution	Circular Convolution
→ If the length of the two sequences $x_1(n)$ & $x_2(n)$ are L_1 & L_2. then the length $\boxed{N = L_1 + L_2 - 1}$	→ If the length of the two sequence $x_1(n)$ & $x_2(n)$ are L_1 & L_2 then the length is $\boxed{N = \max(L_1, L_2)}$
→ $y(n) = x_1(n) * x_2(n)$ $y(n) = \sum_{m=-\infty}^{\infty} x_1(m)\, x_2(n-m)$	→ $y(n) = x_1(n) \textcircled{N} x_2(n)$ $y(n) = \sum_{n=0}^{N-1} x_1(m)\, x_2((n-m))_N$
→ Zero Padding is not necessary. → Linear shifting used	→ Zero Padding is necessary to find the response of the filter. → circular shifting used.

Problem: Find the Linear Convolution of the given sequence using circular Convolution.

$x_1(n) = \{1, 2, 3\}$, $x_2(n) = \{1, -1, 2\}$

Soln: $L_1 = 3$, $L_2 = 3$

Length of the output of Linear Convolution is

$$N = L_1 + L_2 - 1$$

$$= 3+3-1 \Rightarrow \boxed{N=5}$$

In order to obtain the Linear Convolution using circular Convolution, append two zeros to $x_1(n)$ & $x_2(n)$

$x_1(n) = \{1, 2, 3, 0, 0\}$, $x_2(n) = \{1, -1, 2, 0, 0\}$

Using Matrix Multiplication Method.

$$[x_2(n)]_{N\times N} \times [x_1(n)]_{N\times 1} = [y(n)]_{N\times 1}$$

$$\begin{bmatrix} 1 & 0 & 0 & 2 & -1 \\ -1 & 1 & 0 & 0 & 2 \\ 2 & -1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0+0+0 \\ -1+2+0+0+0 \\ 2-2+3+0+0 \\ 0+4-3+0+0 \\ 0+0+6+0+0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \\ 6 \end{bmatrix}$$

$$\boxed{y(n) = \{1, 1, 3, 1, 6\}}$$

Linear Convolution:

$x_1(n) = \{1, 2, 3\}$, $x_2(n) = \{1, -1, 2\}$

$x_2(n)$ \ $x_1(n)$	1	2	3
1	①	2	3
-1	-1	-2	-3
2	2	4	6

①

-1+2 = ①

2-2+3 = ③

4-3 = ①

⑥

$$\boxed{y(n) = \{1, 1, 3, 1, 6\}}$$

FILTERING LONG DURATION SEQUENCE USING OVERLAB - ADD METHOD

Problem: Find the output of the given filter whose input signal and impulse signal are $x(n) = \{1, -1, 1, 2, 1, 0, 1, -4, 3, 2, 1, 0, 1, 1\}$ and $h(n) = \{1, 1, 2, 1\}$ Using overlap-add method.

Soln: Given Data:

The Length of $x(n) = L_{in} = 14$

The Length of $h(n) = M = 4$

Overlap - Add Method:

Step(i): The long duration input sequence is divided into blocks of data with length

$$N = L + M - 1$$

$L \rightarrow$ No. of data from the input $= 2$

$$N = 2 + 4 - 1 \Rightarrow \boxed{N = 5}$$

$$x(n) = \{\underbrace{1, -1}, \underbrace{1, 2}, \underbrace{1, 0}, \underbrace{1, -4}, \underbrace{5, 2}, \underbrace{1, 0}, \underbrace{1, 1}\}$$

$M-1 = 3$ zeros are appended to each data block

$x_1(n) = \{1, -1, 0, 0, 0\}$

$x_2(n) = \{1, 2, 0, 0, 0\}$

$x_3(n) = \{1, 0, 0, 0, 0\}$

$x_4(n) = \{1, -4, 0, 0, 0\}$

$x_5(n) = \{3, 2, 0, 0, 0\}$

$x_6(n) = \{1, 0, 0, 0, 0\}$

$x_7(n) = \{1, 1, 0, 0, 0\}$

Step(ii): Find the output response of each block of data using circular convolution.

$$\boxed{y_1(n) = x_1(n) \text{ Ⓝ } h(n)}$$

$$y_1(n) = \{1, -1, 0, 0, 0\} \text{ Ⓝ } \{1, 1, 2, 1, 0\}$$

Matrix Multiplication Method:

$$[h(n)] \times [x_1(n)] = [y_1(n)]$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+0+0+0+0 \\ 1-1+0+0+0 \\ 2-1+0+0+0 \\ 1-2+0+0+0 \\ 0-1+0+0+0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

$$\boxed{y_1(n) = \{1, 0, 1, -1, -1\}}$$

$$y_2(n) = x_2(n) \text{ Ⓝ } h(n)$$

$$y_2(n) = \{1, 2, 0, 0, 0\} \text{ Ⓝ } \{1, 1, 2, 1, 0\}$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+0+0+0+0 \\ 1+2+0+0+0 \\ 2+2+0+0+0 \\ 1+4+0+0+0 \\ 0+2+0+0+0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 2 \end{bmatrix}$$

$$\boxed{y_2(n) = \{1, 3, 4, 5, 2\}}$$

$y_3(n) = x_3(n) \textcircled{N} h(n)$

$y_3(n) = \{1, 0, 0, 0, 0\} \textcircled{N} \{1, 1, 2, 1, 0\}$

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1+0+0+0+0 \\ 1+0+0+0+0 \\ 2+0+0+0+0 \\ 1+0+0+0+0 \\ 0+0+0+0+0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\boxed{y_3(n) = \{1, 1, 2, 1, 0\}}$$

$y_4(n) = x_4(n) \textcircled{N} h(n)$

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -4 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+0+0+0+0 \\ 1-4+0+0+0 \\ 2-4+0+0+0 \\ 1-8+0+0+0 \\ 0-4+0+0+0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ -3 \\ -2 \\ -7 \\ -4 \end{bmatrix}$$

$$\boxed{y_4 = \{1, -3, -2, -7, -4\}}$$

$y(5) = x_5(n) \textcircled{N} h(n)$

$y_5(n) = \{3, 2, 0, 0, 0\} \textcircled{N} \{1, 1, 2, 1, 0\}$

$$= \begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3+0+0+0+0 \\ 3+2+0+0+0 \\ 6+2+0+0+0 \\ 3+4+0+0+0 \\ 0+2+0+0+0 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 \\ 5 \\ 8 \\ 7 \\ 2 \end{bmatrix}$$

$$\boxed{y_5(n) = \{3, 5, 8, 7, 2\}}$$

$y_6(n) = x_6(n) \textcircled{N} h(n)$

$y_6(n) = \{1, 0, 0, 0, 0\} \textcircled{N} \{1, 1, 2, 1, 0\}$

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+0+0+0+0 \\ 1+0+0+0+0 \\ 2+0+0+0+0 \\ 1+0+0+0+0 \\ 0+0+0+0+0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\boxed{y_6(n) = \{1, 1, 2, 1, 0\}}$$

$y_7(n) = x_7(n) \textcircled{N} h(n)$

$y_7(n) = \{1, 1, 0, 0, 0\} \textcircled{N} \{1, 1, 2, 1, 0\}$

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+0+0+0+0 \\ 1+1+0+0+0 \\ 2+1+0+0+0 \\ 1+2+0+0+0 \\ 0+1+0+0+0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \\ 1 \end{bmatrix}$$

$$\boxed{y_7(n) = \{1, 2, 3, 3, 1\}}$$

Step(iii) :

The last 3(m-1) data points from each output block must be overlapped & added to the First 3 data of the Success Blocks

$y_1(n)$: 1 0 1 -1 -1

$y_2(n)$: 1 3 4 5 2

$y_3(n)$: 1 1 2 1 0

$y_4(n)$: 1 -3 -2 -7 -4

$y_5(n)$: 3 5 8 7 2

$y_6(n)$: 1 1 2 1 0

$y_7(n)$: 1 2 3 3 1

$$\boxed{y(n) = \{ 1\ 0\ 2\ 2\ 4\ 6\ 5\ -2\ 1\ -2\ 5\ 8\ 5\ 3\ 3\ 3\ 1 \}}$$

Problem: ②

Find the output of the given filter whose input signal & impulse signal are $x(n) = \{1, -1, 1, 2, 1, 0, 1, -4, 3, 2, 1, 0, 1, 1\}$ $h(n) = \{1, 1, 2, 1\}$ using overlap-save Method.

Soln: Given: The length of input $x(n) = 14 = L_{in}$

The length of impulse $h(n) = 4 = M$

Step 1: The input sequence is divided into blocks of data with length $\boxed{L + M - 1 = N}$

Let's: $L = 2$, $M - 1 = 3$

$N = 2 + 4 - 1 \Rightarrow \boxed{N = 5}$

$x(n) = \{\underbrace{1, -1}, 1, 2, 1, 0, 1, -4, 3, 2, 1, 0, 1, 1\}$

Input sequence is divided in Blocks of data: Block = 5

$x_1(n) = \{\underbrace{0, 0, 0}_{\substack{\text{Zeros} \\ m-1 = 3}}, \overbrace{1, -1}^{L = 2 : \text{Data}}\}$

$x_2(n) = \{0, 1, -1, \underbrace{1, 2}_{\text{Next Data}}\}$

$x_3(n) = \{-1, 1, 2, \underbrace{1, 0}\}$

$x_4(n) = \{2, 1, 0, \underbrace{1, -4}\}$

$x_5(n) = \{0, 1, -4, 3, 2\}$

$x_6(n) = \{-4, 3, 2, \underbrace{1, 0}\}$

$x_7(n) = \{2, 1, 0, \underbrace{1, 1}\}$

$x_8 = \{0, 1, \underbrace{1, 0, 0}\}$

$x_9 = \{1, 0, \underbrace{0, 0, 0}\}$

Step 2: To find the circular convolution for each data block with an Impulse response

$$y_1(n) = x_1(n) \textcircled{N} h(n)$$

$$y_1(n) = \{0, 0, 0, 1, -1\} \textcircled{N} \{1, 1, 2, 1, 0\}$$

$$[h(n)]\,[x_1(n)] = [y_1(n)]$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0+0+0+2-1 \\ 0+0+0+1-2 \\ 0+0+0+0-1 \\ 0+0+0+1+0 \\ 0+0+0+1-1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\boxed{y_1(n) = \{1, -1, -1, 1, 0\}}$$

$$y_2(n) = x_2(n) \textcircled{N} h(n)$$

$$[h(n)]\,[x_2(n)] = y_2(n)$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0+0-1+2+2 \\ 0-1+0+1+4 \\ 0+1-1+0+2 \\ 0+2-1+1+0 \\ 0+1-2+1+2 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 \\ 6 \\ 2 \\ 2 \\ 2 \end{bmatrix}$$

$$\boxed{y_2(n) = \{3, 6, 2, 2, 2\}}$$

$$y_3(n) = \{-1, 1, 2, 1, 0\} \textcircled{N} \{1, 1, 2, 1, 0\}$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1+0+1+4+0 \\ -1+1+0+1+0 \\ -2+1+2+0+0 \\ -1+2+2+1+0 \\ 0+1+4+1+0 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 \\ 1 \\ 1 \\ 4 \\ 6 \end{bmatrix}$$

$$\boxed{y_3(n) = \{4, 1, 1, 4, 6\}}$$

$y_4(n) = \{2, 1, 0, 1, -4\} \textcircled{N} \{1, 1, 2, 1, 0\}$

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 2+0+0+2-4 \\ 2+1+0+1-8 \\ 4+1+0+0-4 \\ 2+2+0+1+0 \\ 0+1+0+1-4 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ -4 \\ 1 \\ 5 \\ -2 \end{bmatrix}$$

$$\boxed{y_4(n) = \{0, -4, 1, 5, -2\}}$$

$y_5(n) = x_5(n) \textcircled{N} h(n)$

$y_5(n) = \{0, 1, -4, 3, 2\} \textcircled{N} \{1, 1, 2, 1, 0\}$

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0+0-4+6+2 \\ 0+1+0+3+4 \\ 0+1-4+0+2 \\ 0+2-4+3+0 \\ 0+1-8+3+2 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 \\ 8 \\ -1 \\ 1 \\ -2 \end{bmatrix}$$

$$\boxed{y_5(n) = \{4, 8, -1, 1, -2\}}$$

$y_6(n) = \{-4, 3, 2, 1, 0\} \textcircled{N} \{1, 1, 2, 1, 0\}$

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ 3 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4+0+2+2+0 \\ -4+3+0+1+0 \\ -8+3+2+0+0 \\ -4+6+2+1+0 \\ 0+3+4+1+0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 0 \\ -3 \\ 5 \\ 8 \end{bmatrix}$$

$$\boxed{y_6(n) = \{0, 0, -3, 5, 8\}}$$

$y_7(n) = \{2, 1, 0, 1, 1\}$ Ⓝ $\{1, 1, 2, 1, 0\}$

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+0+0+2+1 \\ 2+1+0+1+2 \\ 4+1+0+0+1 \\ 2+2+0+1+0 \\ 0+1+0+1+\text{[illegible]} \end{bmatrix} \Rightarrow \begin{bmatrix} 5 \\ 6 \\ 6 \\ 5 \\ 3 \end{bmatrix}$$

$$\boxed{y_7(n) = \{5, 6, 6, 5, 3\}}$$

$y_8(n) = \{0, 1, 1, 0, 0\}$ Ⓝ $\{1, 1, 2, 1, 0\}$

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0+0+1+0+0 \\ 0+1+0+0+0 \\ 0+1+1+0+0 \\ 0+2+1+0+0 \\ 0+1+2+0+0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}$$

$$\boxed{y_8(n) = \{1, 1, 2, 3, 3\}}$$

$y_9(n) = \{1, 0, 0, 0, 0\}$ Ⓝ $\{1, 1, 2, 1, 0\}$

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+0+0+0+0 \\ 1+0+0+0+0 \\ 2+0+0+0+0 \\ 1+0+0+0+0 \\ 0+0+0+0+0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\boxed{y_9(n) = \{1, 1, 2, 1, 0\}}$$

Step(iii)

The M-1 : (3) → First 3 data must be discarded Fro output block and finally Combine the output blocks.

$y_1(n) = \{\underbrace{1, -1, -1}_{\text{discarded}}, 1, 0\}$

$y_2(n) = \{\underbrace{3, 6, 2}, 2, 2\}$

$y_3(n) = \{\underbrace{4, 1, 1}, 4, 6\}$

$y_4(n) = \{\underbrace{0, -4, 1}, 5, -2\}$

$y_5(n) = \{\underbrace{4, 8, -1}, 1, -2\}$

$y_6(n) = \{\underbrace{0, 0, -3}, 5, 8\}$

$y_7(n) = \{\underbrace{5, 6, 6}, 5, 3\}$

$y_8(n) = \{\underbrace{1, 1, 2}, 3, 3\}$

$y_9(n) = \{\underbrace{1, 1, 2}, 1, 0\}$

Ans:

$$\boxed{\begin{aligned} y(n) = \{&1, 0, 2, 2, 4, 6, 5, -2, 1, -2 \\ &5, 8, 5, 3, 3, 3, 1, 0\} \end{aligned}}$$

Using Linear Convolution Construct $y(n) = x(n) * h(n)$ For the Sequence $x(n) = \{1, 2, -1, 2, 3, -2, -3, -1, 1, 1, 2, -1\}$ and $h(n) = \{1, 2\}$ Compare the result by solving the problem using overlap add method & overlap save method.

Condition

$$\boxed{\begin{array}{c} L > M \\ L = N-(M-1) \end{array}}$$

Soln: $L = 3$, $N = L+M-1$

$\boxed{M = 2}$ $\quad = 3+2-1 \Rightarrow \boxed{N = 4}$

$$x(n) = \{\underbrace{1, 2, -1}, \underbrace{2, 3, -2}, \underbrace{-3, -1, 1}, \underbrace{1, 2, -1}\}$$

Overlap Save method:

$$\boxed{h(n) = \{1, 2, 0, 0\}}$$

$$x_1(n) = \{0, 1, 2, \textcircled{-1}\}$$
$$x_2(n) = \{-1, 2, 3, \textcircled{-2}\}$$
$$x_3(n) = \{-2, -3, -1, \textcircled{1}\}$$
$$x_4(n) = \{1, 1, 2, \textcircled{-1}\}$$
$$x_5(n) = \{-1, 0, 0, 0\}$$

$$y_1(n) = x_1(n) \textcircled{N} h(n)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 2 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0+0+0-2 \\ 0+1+0+0 \\ 0+2+2+0 \\ 0+0+4-1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 4 \\ 3 \end{bmatrix}$$

$$\boxed{y_1(n) = \{-2, 1, 4, 3\}}$$

$$y_2(n) = x_2(n) \textcircled{N} h(n)$$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -1+0+0-4 \\ -2+2+0+0 \\ 0+4+3+0 \\ 0+0+6-2 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 7 \\ 4 \end{bmatrix}$$

$$\boxed{y_2(n) = \{-5, 0, 7, 4\}}$$

$y_3(n) = \{ x_3(n) \textcircled{N} h(n) \}$

$$= \begin{bmatrix} 1 & 0 & 0 & 2 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -2+0+0+2 \\ -4-3+0+0 \\ 0-6-1+0 \\ 0+0-2+1 \end{bmatrix} = \begin{bmatrix} 0 \\ -7 \\ -7 \\ -1 \end{bmatrix}$$

$$\boxed{y_3(n) = \{0, -7, -7, -1\}}$$

$y_4(n) = x_4(n) \textcircled{N} h(n)$

$$= \begin{bmatrix} 1 & 0 & 0 & 2 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1+0+0-2 \\ 2+1+0+0 \\ 0+2+2-0 \\ 0+0+4-1 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 \\ 3 \\ 4 \\ 3 \end{bmatrix}$$

$$\boxed{y_4(n) = \{-1, 3, 4, 3\}}$$

$y_5(n) = x_5(n) \textcircled{N} h(n)$

$$= \begin{bmatrix} 1 & 0 & 0 & 2 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1+0+0+0 \\ -2+0+0+0 \\ 0+0+0+0 \\ 0+0+0+0 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{y_5(n) = \{-1, -2, 0, 0\}}$$

$y_0(n)$: $\boxed{-2}$ 1 4 3

$y_1(n)$: ———— $\boxed{-5}$ 0 7 4

$y_2(n)$: ———— $\boxed{0}$ -7 -7 -1

$y_3(n)$: ———— $\boxed{-1}$ 3 4 3

discard

$y_4(n)$: ———— $\boxed{-1}$ -2 0 0

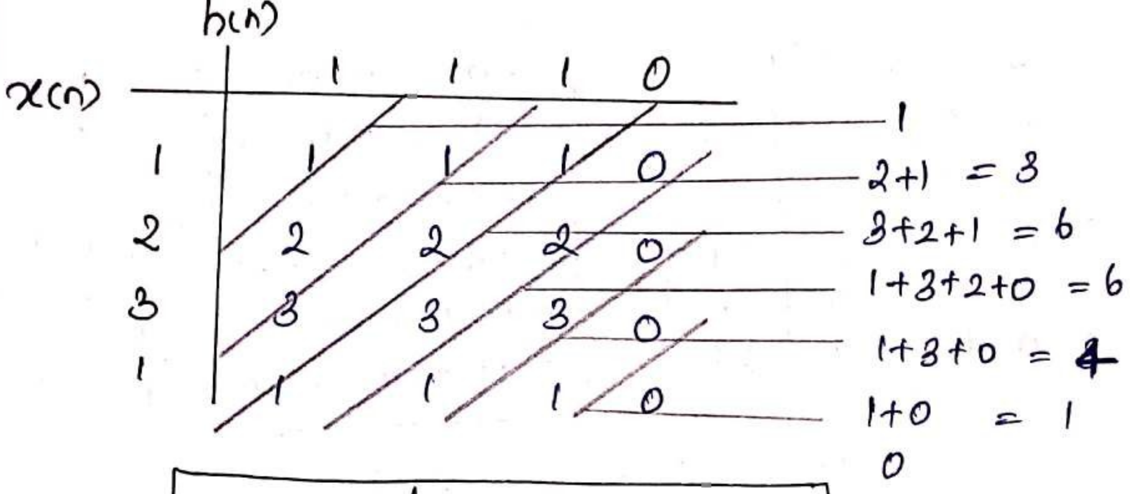
$$\boxed{y(n) = \{1, 4, 3, 0, 7, 4, -7, -7, -1, 3, 4, 3, -2, 0, 0\}}$$

Problem:

Determine the output response $y(n)$ if $h(n) = \{1, 1, 1\}$ $x(n) = \{1, 2, 3, 1\}$ by using linear convolution, circular convolution & circular convolution with zero padding.

Soln: Linear Convolution:

$x(n) = \{1, 2, 3, 1\}$, $h(n) = \{1, 1, 1\}$



$$\boxed{y(n) = \{1, 3, 6, 6, 4, 0\}}$$

Number of samples in linear convolution is $\boxed{L+M-1}$

$L = 4$, $M = 3$, $\boxed{4+3-1 = 6}$

Circular Convolution:

$x(n) = \{1, 2, 3, 1\}$, $h(n) = \{1, 1, 1, 0\}$

Using Matrix Method $y(n) = x(n) \textcircled{N} h(n)$

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+0+3+1 \\ 1+2+0+1 \\ 1+2+3+0 \\ 0+2+3+1 \end{bmatrix} \Rightarrow \begin{bmatrix} 5 \\ 4 \\ 6 \\ 6 \end{bmatrix}$$

$$\boxed{y(n) = \{5, 4, 6, 6\}}$$

By comparing circular convolution output with that of linear convolution we find the first 2 point (M-1) are aliased.

Last two data points are added to first two data points in Linear Convolution are added to first two data points as shown below $1+4=5$, $3+1=4$

Circular Convolution with zero padding:

Add $(M-1)$ zeros with $x(n)$ and $(L-1)$ zeros with $h(n)$, $x(n) = \{1, 2, 3, 1, 0, 0\}$, $h(n) = \{1, 1, 1, 0, 0, 0\}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+0+0+0+0+0 \\ 1+2+0+0+0+0 \\ 1+2+3+0+0+0 \\ 0+2+3+1+0+0 \\ 0+0+3+1+0+0 \\ 0+0+0+1+0+0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 3 \\ 6 \\ 6 \\ 4 \\ 1 \end{bmatrix}$$

$$\boxed{y(n) = \{1, 3, 6, 6, 4, 1\}}$$

Characteristics of Practical frequency Selective Filters - Characteristics of Commonly Used analog Filters - Butterworth Filters, Chebyshev Filters. - Design of IIR Filters from analog filters (LPF, HPF, BPF, BRF) - Approximation of derivatives, Impulse Invariance Method - Bilinear Transformation frequency transformation in the analog domain - Structure of IIR Filter - Direct form I, direct form II - Cascade, Parallel realizations.

1. Characteristics of practical frequency selective Filters:

→ Depending upon the rejection and passing of the range of frequencies, the filters are classified as Lowpass, High pass, Bandpass, Band rejected Filters.

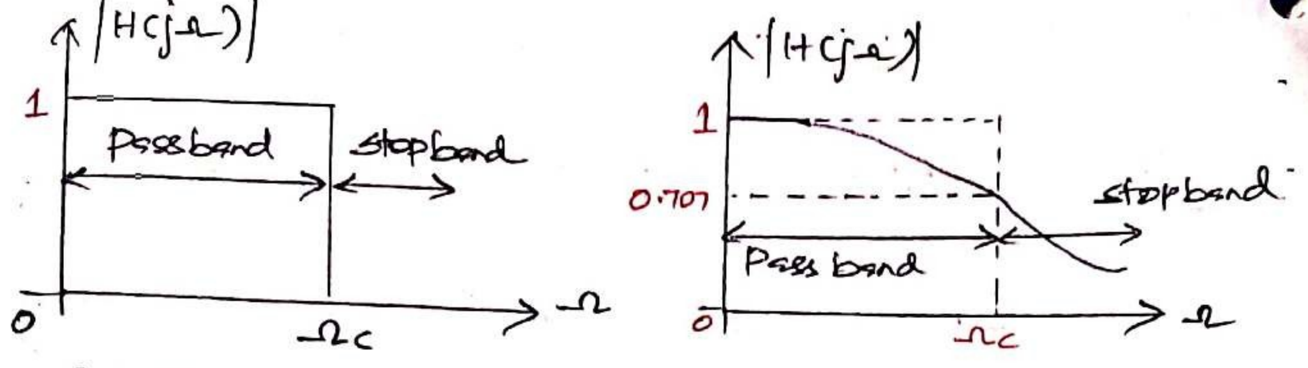
(i) Low pass Filter

→ Allows all the frequencies less than cut-off frequency Ω_c to pass through.

→ It Blocks all the frequencies higher than Ω_c

Ideal frequency response

$$\left. |H(j\Omega)| \right\} = \begin{cases} 1 & \text{for } |\Omega| < \Omega_c \\ 0 & \text{for } |\Omega| > \Omega_c \end{cases}$$



(a) Ideal lowpass filter (b) practical lowpass filter

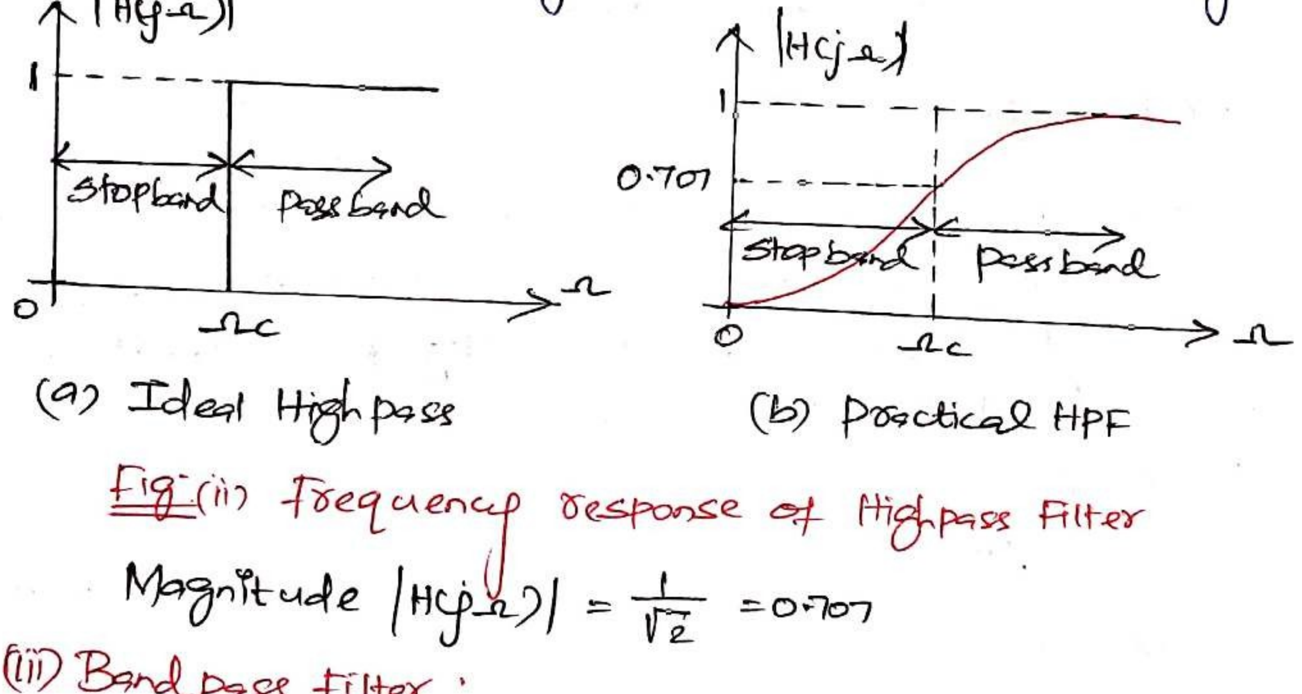
Fig(i) Frequency response of Lowpass Filters.

Magnitude of the output] $H(j\Omega) = \frac{1}{\sqrt{2}} = 0.707$

(ii) High pass Filter:

→ All frequencies higher than Ω_c (or) cutoff frequency

⇒ Block all frequency lower than cut off frequency (Ω_c)



(a) Ideal High pass (b) Practical HPF

Fig (ii) Frequency response of Highpass Filter

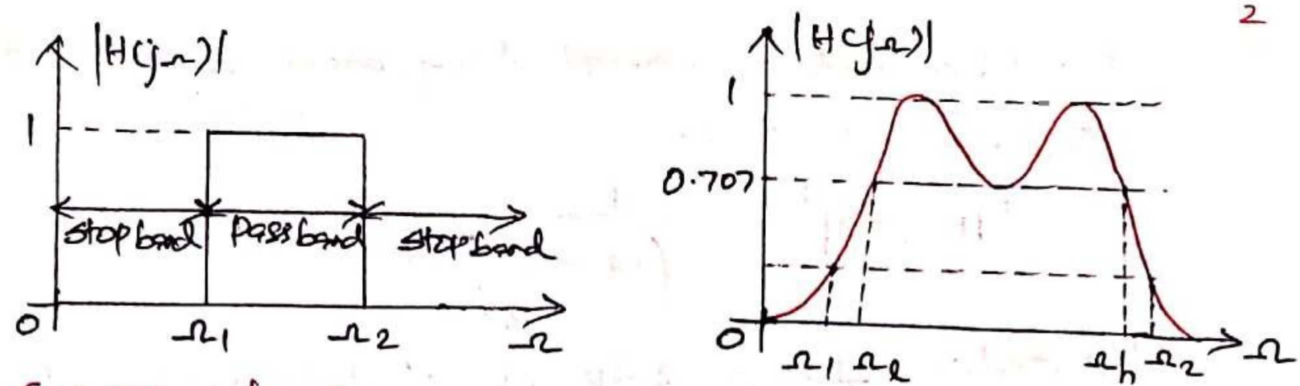
Magnitude $|H(j\Omega)| = \frac{1}{\sqrt{2}} = 0.707$

(iii) Band pass Filter:

→ This filter passes certain band of frequencies from Ω_1 to Ω_2.

→ Reject all other frequencies.

frequency $|H(j\Omega)| = \frac{1}{\sqrt{2}} = 0.707$

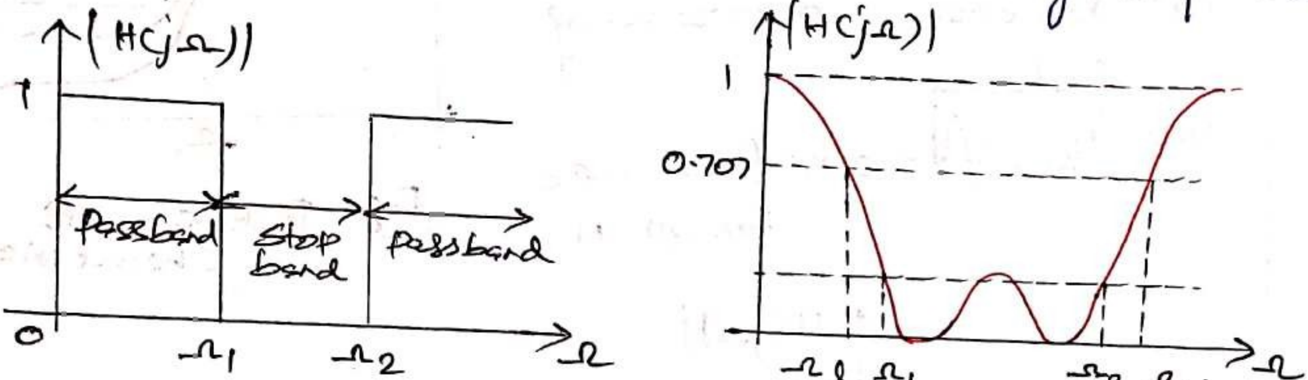


(a) Ideal bandpass filter (b) Practical BPF

Fig:(iii) Frequency response of BPF

4. Band Reject (or) Notch filter:

→ This filter rejects all the frequencies from Ω_1 to Ω_2 and allows all other remaining frequencies.



(a) Ideal band reject filter . (b) Practical BRF

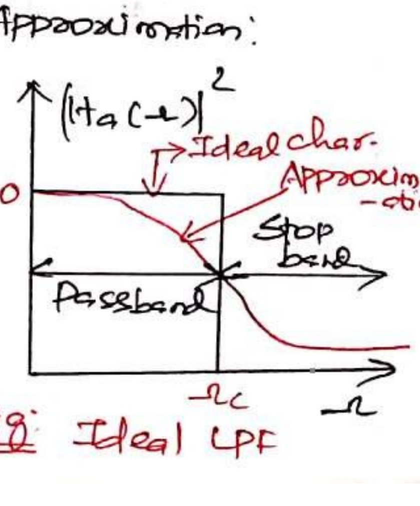
(Fig iv) : Frequency response of BRF

Frequency $|H(j\Omega)| = \frac{1}{\sqrt{2}} = 0.707$

2. Characteristics of Commonly used Analog filters:

Analog filter Design Using Butterworth Approximation:

Why Filter Approximation ?



→ The Ideal LPF response is not physically realizable. hence the response is approximated with the help of Standard function such as Butterworth, chebyshev, Elliptic, etc

Fig: Ideal LPF

The Magnitude squared frequency response of the Butterworth filter.

$$|H_a(\Omega)|^2 = \frac{1}{\left(1+\frac{\Omega}{\Omega_c}\right)^{2N}}$$

$N \Rightarrow$ order, $\Omega_c \Rightarrow$ -3dB cut-off frequency of the filter monotonically reducing Magnitude response.

Butterworth characteristic:

(i) Characteristic is close to Ideal response when order is increased.

(ii) Response is monotonically reducing.

(iii) $|H_a(\Omega)|^2 = 0.5$ for $\Omega = \Omega_c$ For all N.

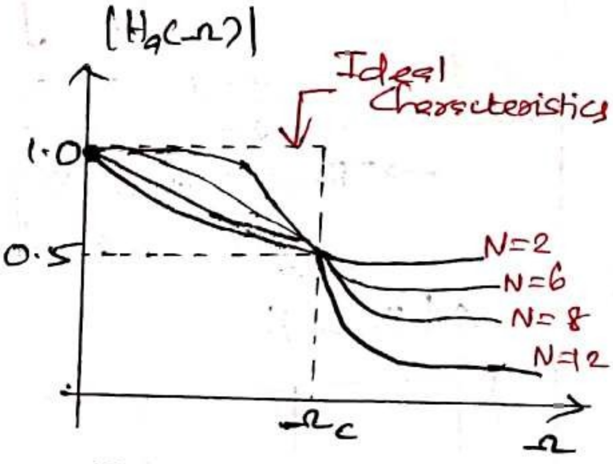


Fig: Butterworth Characteristic

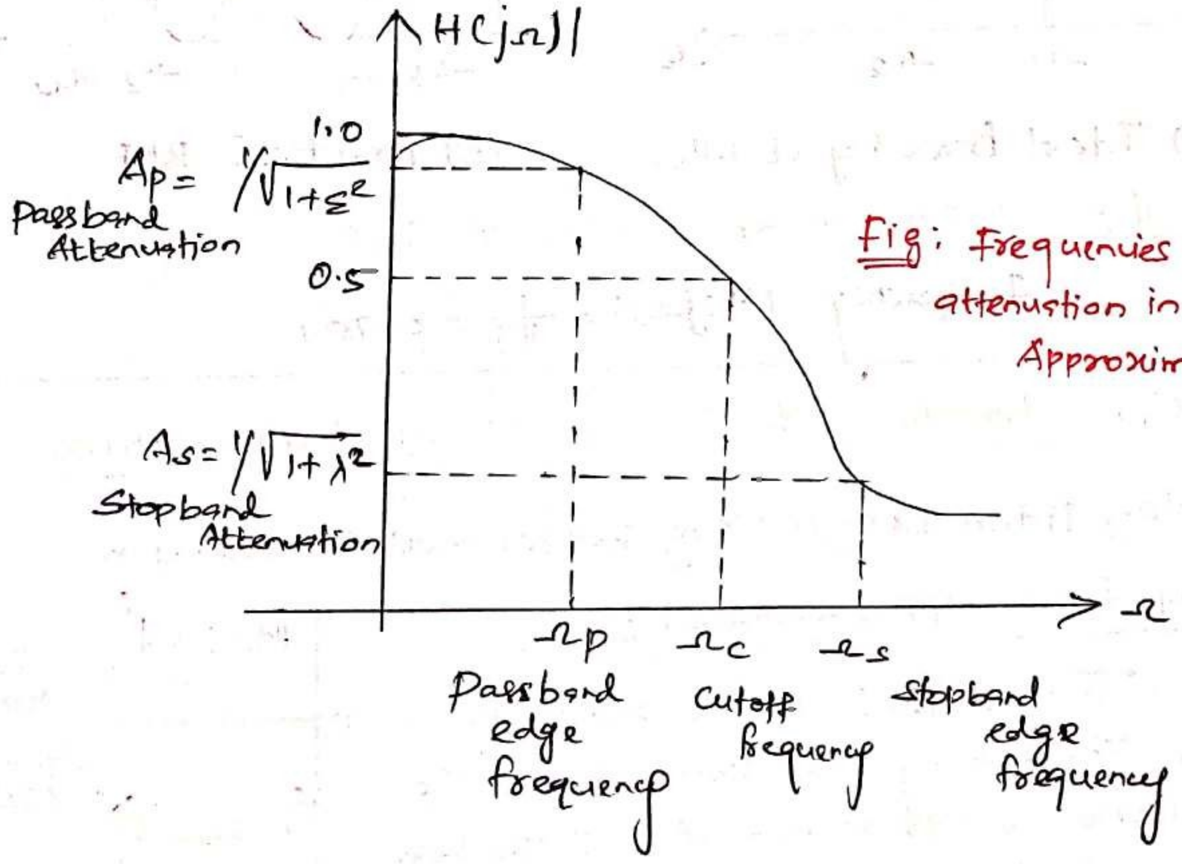


Fig: Frequencies and attenuation in Butterworth Approximation.

Design Steps For Butterworth filter:

Step 1: From the given specification find the order of filter 'N' using the following formula.

$$N \geq \frac{\log\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log\left(\frac{\Omega_s}{\Omega_p}\right)} \quad \text{(or)} \quad N \geq \frac{\log(\lambda/\varepsilon)}{\log(\Omega_s/\Omega_p)}$$

where,

$$\lambda = \sqrt{10^{0.1\alpha_s}-1}$$

$$\varepsilon = \sqrt{10^{0.1\alpha_p}-1}$$

Step 2: Find the Transfer function $H(s)$ for $\boxed{\Omega_c = 1 \text{ rad/sec}}$ from the table for value of N.

List of Butterworth polynomial:

Order of the Filter(N)	Transfer function H(s)
N=1	$H(s) = \frac{1}{s+1}$
N=2	$H(s) = \frac{1}{s^2+\sqrt{2}\,s+1}$
N=3	$H(s) = \frac{1}{(s+1)(s^2+s+1)}$
N=4	$H(s) = \frac{1}{(s^2+0.76537s+1)(s^2+1.8477s+1)}$
N=5	$H(s) = \frac{1}{(s+1)(s^2+0.61803s+1)(s^2+1.61803s+1)}$

Step 3: Calculate the Value of cut-off frequency (Ω_c).

$$\Omega_c = \frac{\Omega_p}{\left(10^{0.1\alpha_p}-1\right)^{1/2N}} \quad (\text{or}) \quad \Omega_c = \frac{\Omega_p}{(\epsilon)^{1/N}}$$

Step 4: Find the Transfer function $H_a(s)$ for the Value of Ω_c.

$$H_a(s) = H(s)\Big|_{s \to \frac{s}{\Omega_c}}$$

Step 5: Find $H(z)$

(a) For Bilinear Transformation:

$$H(z) = H_a(s)\Big|_{s \to \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

The prewarping Analog frequencies is obtained in Bilinear Transformation is.

$$\Omega_s = \frac{2}{T}\tan\left(\frac{\omega_s}{2}\right)$$
$$\Omega_p = \frac{2}{T}\tan\left(\frac{\omega_p}{2}\right)$$

(b) For Impulse Invarient Method:

$H(z)$ Can be obtained as, $H(s) = \sum_{i=1}^{N} \frac{c}{s+p_i}$ will be Transformed to

$$H(z) = \sum_{i=1}^{N} \frac{c}{1-e^{-p_iT}z^{-1}}$$

$$\frac{1}{s+p_i} \xrightarrow{} \frac{1}{1-e^{-p_iT}z^{-1}}$$
Transformed

In impulse Invarient method, the analog frequency is obtained as $\Omega_s = \frac{\omega_s}{T}$, $\Omega_p = \frac{\omega_p}{T}$

Problem:

Design a digital Butterworth Filter Satisfying the following specifications:

$$0.8 \le |H(e^{j\omega})| \le 1 \quad \text{for } 0 \le \omega \le 0.2\pi$$

$$|H(e^{j\omega})| \le 0.2 \quad \text{for } 0.6\pi \le \omega \le \pi$$

Assume T = 1 sec. Apply Bilinear Transformation.

Soln: Given: $\omega p = 0.2\pi, \ \omega_s = 0.6\pi$

To find Prewarping Analog frequency for Bilinear Transform

$$\Omega p = \frac{2}{T} \tan \frac{\omega p}{2}$$

$$= \frac{2}{1} \tan \frac{0.2\pi}{2}$$

$$\boxed{\Omega p = 0.6498 \text{ rad/sec}}$$

$$\Omega s = \frac{2}{T} \tan \frac{\omega s}{2}$$

$$= \frac{2}{1} \tan \frac{0.6\pi}{2}$$

$$\boxed{\Omega s = 2.752 \text{ rad/sec}}$$

To find λ, ε.

$$\frac{1}{\sqrt{1+\varepsilon^2}} = 0.8$$

$$\frac{1}{0.8} = \sqrt{1+\varepsilon^2}$$

Take squaring on both sides

$$\left(\frac{1}{0.8}\right)^2 = \left(\sqrt{1+\varepsilon^2}\right)^2$$

$$1.5625 - 1 = \varepsilon^2$$

$$\frac{1}{\sqrt{1+\lambda^2}} = 0.2$$

$$\frac{1}{0.2} = \sqrt{1+\lambda^2}$$

Take squaring on both sides

$$\left(\frac{1}{0.2}\right)^2 = \left(\sqrt{1+\lambda^2}\right)^2$$

$$25 - 1 = \lambda^2$$

$0.5625 = \varepsilon^2$ | $24 = \lambda^2$

$\sqrt{0.5625} = \varepsilon$ | $\sqrt{24} = \lambda$

$\boxed{\varepsilon = 0.75}$ | $\boxed{\lambda = 4.8989}$

Step 1 : To Find order of Filter N.

$$N \geq \frac{\log(\lambda/\varepsilon)}{\log(\Omega_s/\Omega_p)}$$

$$N \geq \frac{\log\left(\frac{4.8989}{0.75}\right)}{\log\left(\frac{2.752}{0.6498}\right)}$$

$$\boxed{N = 2}$$

Step 2 : To find Analog Transfer Function H(s)

for $N = 2$, $\boxed{H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}}$ ... Refer Butterworth polynomial Table.

Step 3 : To Find cut-off frequency Ω_c

$$\Omega_c = \frac{\Omega_p}{(\varepsilon)^{1/N}}$$

$$= \frac{0.6498}{(0.75)^{1/2}}$$

$$\boxed{\Omega_c = 0.75}$$

Step 4 : To Find $H_a(s)$ for $\Omega_c = 0.75$

$$H_a(s) = H(s)\Big|_{s \to \frac{s}{\Omega_c}}$$

$$= \frac{1}{\left(\frac{s}{0.75}\right)^2 + \sqrt{2}\left(\frac{s}{0.75}\right) + 1}$$

$$= \frac{1}{\frac{s^2}{(0.75)^2} + \frac{\sqrt{2}\,s + 0.75}{(0.75)}}$$

$$= \frac{(0.75)^3}{(0.75)s^2 + (0.75)^2\sqrt{2}\,s + (0.75)^3}$$

$$= \frac{(0.75)^{\cancel{3}}}{\cancel{0.75}\left[s^2 + 0.75\sqrt{2}\,s + (0.75)^2\right]}$$

$$\boxed{H_a(s) = \frac{0.5625}{s^2 + 1.06\,s + 0.5625}}$$

Step 5: To Find H(z)

For Bilinear Transformation

$$H(z) = H_a(s)\Big|_{s \rightarrow \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{0.5625}{s^2 + 1.06\,s + 0.5625}\Bigg|_{s \rightarrow \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{0.5625}{\left[\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]^2 + 1.06\left[\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right] + 0.5625}$$

To Find ω_c

$$\boxed{\omega_c = \frac{2\pi f_c}{f_s}}$$

$$= \frac{2\pi \times 1000}{10,000}$$

$\because f_s = 10\,kHz$
$= 10 \times 10^3$
$f_c = 1\,kHz$
$= 1 \times 10^3$

$$\boxed{\omega_c = \frac{\pi}{5}}$$

Step 1: To find order of the Filter N

Given: $\boxed{N = 2}$ order of the Filter

Step 2: To Find H(s).

For $N=2$: the 2^{nd} order Transfer function

$$\boxed{H(s) = \frac{1}{s^2+\sqrt{2}s+1}}$$

Step 3: To Find cutoff frequency Ω_c

$$\Omega_c = \frac{2}{T}\tan\left(\frac{\omega_c}{2}\right)$$

$$= \frac{2}{1}\tan\left(\left(\frac{\pi}{5}\right)/_2\right)$$

$$\boxed{\Omega_c = 0.649}$$

Step 4: To Find $H_a(s)$

$$H_a(s) = H(s)\Big|_{s \to \frac{s}{\Omega_c}}$$

$$= \frac{1}{s^2+\sqrt{2}s+1}\Big|_{s \to \frac{s}{0.649}}$$

$$= \frac{1}{\left(\frac{s}{0.649}\right)^2 + \sqrt{2}\left(\frac{s}{0.649}\right) + 1}$$

$$= \frac{1}{\frac{s^2}{(0.649)^2} + \frac{\sqrt{2}s}{0.649} + 1}$$

$$= \frac{1}{\frac{s^2 + \sqrt{2}s(0.649) + (0.649)^2}{(0.649)^2}}$$

$$\boxed{H_a(s) = \frac{(0.649)^2}{s^2 + 0.917s + (0.649)^2}}$$

Step 5: To Find H(z)

For Bilinear Transform $H(z) = H_a(s)\Big|_{s \to \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$

$$= \frac{(0.649)^2}{s^2 + 0.917s + (0.649)^2}\Bigg|_{s \to \frac{2}{1}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{(0.649)^2}{\left[\frac{2}{1}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]^2 + 0.917\left[\frac{2}{1}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right] + (0.649)^2}$$

$$= \frac{(0.649)^2}{\frac{4(1-z^{-1})^2}{(1+z^{-1})^2} + 1.835\frac{(1-z^{-1})}{(1+z^{-1})} + (0.649)^2}$$

$$= \frac{0.421}{\dfrac{4(1-z^{-1})^2+1.835(1-z^{-1})(1+z^{-1})+0.421(1+z^{-1})^2}{(1+z^{-1})^2}}$$

$$= \frac{0.421\,(1+z^{-1})^2}{4(1-2z^{-1}+z^{-2})+1.835(1-z^{-2})+0.421(1+2z^{-1}+z^{-2})}$$

$$= \frac{0.421(1+z^{-1})^2}{4-8z^{-1}+4z^{-2}+1.835-1.835z^{-2}+0.421+0.842z^{-1}+0.421z^{-2}}$$

$$\boxed{H(z) = \frac{0.421(1+z^{-1})^2}{6.256-7.158z^{-1}+2.586z^{-2}}}$$

Problem 3:

Using the bilinear Transform design a High Pass Filter monotonic in passband with cut-off frequency 1000 Hz and down 10dB at 350 Hz. the sampling frequency is 5000 Hz.

Given:
Passband Cut-off frequency $f_p = 1000$ Hz
Sampling frequency $f_{SF} = 5000$ Hz
Stopband attenuation $\alpha_s = 10$ dB
Stopband cutoff frequency $f_s = 350$ Hz

An passband attenuation 3dB achieved at cutoff freq f_c.
$\alpha_p = 3$dB.

$\alpha_p = 3$dB $\quad f_p = 1000$ Hz

$\alpha_s = 10$dB $\quad f_s = 350$ Hz

$T = \frac{1}{f_{SF}} = \frac{1}{5000} \quad T = 0.0002 \Rightarrow \boxed{T = 2\times10^{-4}\text{ sec}}$

Discrete Time frequency. $\omega p, \omega s$.

$$\omega p = 2\pi f_p, \quad \omega_s = 2\pi f_s$$

To Find f_P, f_s

$$f_p = \frac{F_P}{F_{SF}} = \frac{1000}{5000} \Rightarrow \boxed{f_p = 0.2}$$

$$f_s = \frac{F_s}{F_{SF}} = \frac{850}{5000} \Rightarrow \boxed{f_s = 0.07}$$

$$W_P = 2\pi f_P = 2\pi \times 0.2 = \boxed{\omega p = 0.4\pi}$$

$$\omega_s = 2\pi f_s = 2\pi \times 0.07 \Rightarrow \boxed{\omega_s = 0.14\pi}$$

Thus, the specification are,

$\alpha_p = 8\,dB \quad \omega p = 0.4\pi$

$\alpha_s = 10\,dB \quad \omega s = 0.14\pi$

Step 1: To find the order of the Filter N.

$$N \geq \frac{\log\sqrt{\dfrac{(10^{0.1\alpha_s})-1}{(10^{0.1\alpha_P})-1}}}{\log\left(\dfrac{\Omega_s}{\Omega_p}\right)}$$

To Find Prewarping Analog frequency. $\Omega p, \Omega s$
(Bilinear Transform),

$$\Omega P = \frac{2}{T}\tan\left(\frac{\omega p}{2}\right) = \frac{2}{2\times10^{-4}}\tan\left(\frac{0.4\pi}{2}\right) = \boxed{7265.425 \text{ rad/sec}}$$

$$\Omega s = \frac{2}{T}\tan\left(\frac{W_s}{2}\right) = \frac{2}{2\times10^{-4}}\tan\left(\frac{0.14\pi}{2}\right) = \boxed{2235.264 \text{ rad/sec}}$$

Order of the Filter N

$$N \geq \frac{\log\sqrt{\frac{(10^{0.1\alpha_s})-1}{(10^{0.1\alpha_p})-1}}}{\log\left(\frac{\Omega_s}{\Omega_p}\right)} \Rightarrow N \geq \frac{\log\sqrt{\frac{(10^{0.1\times10})-1}{(10^{0.1\times3})-1}}}{\log\left(\frac{2235.264}{7265.425}\right)}$$

$$N \geq \frac{\log\sqrt{\frac{10-1}{1.9952-1}}}{\log(0.3076)} \Rightarrow N \geq \frac{\log(3.00)}{\log(0.3076)}$$

$$N \geq 0.9891$$

Order of the Filter $\boxed{N=1}$

Step 2: To Find Transfer Function H(s)

For $\underline{N=1}$, $\boxed{H(s)=\frac{1}{s+1}}$ Refer Butterworth Polynomial table

Step 3: To Find the cut off frequency Ω_c

The High pass filter for $\Omega_c = \Omega_p = 7265.42$ rad/sec

$$\boxed{\Omega_c = 7265.42 \text{ rad/sec}}$$

Step 4: Find the Transfer function of Ha(s) for the Value of Ω_c

$$H_a(s) = H(s)\Big|_{s\to\frac{s}{\Omega_c}}$$

$$= \frac{1}{s+1}\Big|_{s\to\frac{s}{7265.42}}$$

$$\frac{1}{\frac{s}{7265.42}+1} \Rightarrow \frac{1}{\frac{s+7265.42}{s}}$$

$$\boxed{H_a(s) = \frac{s}{s+7265.42}}$$

Step 5: Find H(z)

To apply Bilinear Transform

$$H(z) = H_a(s)\Big|_{s \to \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

$$= \frac{s}{s+7265.42}\Bigg|_{s \to \frac{2}{2\times10^{-4}}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

$$= \frac{\frac{\cancel{2}}{\cancel{2}\times10^{-4}}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}{\left(\frac{\cancel{2}}{\cancel{2}\times10^{-4}}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]\right)+7265.42} \Rightarrow \frac{10^4\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}{10^4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+7265.42}$$

$$= \frac{10000\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}{\frac{10000(1-z^{-1})+7265.42(1+z^{-1})}{(1+z^{-1})}}$$

$$= \frac{10000\left(\frac{1-z^{-1}}{\cancel{1+z^{-1}}}\right)\times\cancel{(1+z^{-1})}}{10000(1-z^{-1})+7265.42(1+z^{-1})}$$

$$= \frac{10000(1-z^{-1})}{10000-10000z^{-1}+7265.42+7265.42z^{-1}}$$

$$= \frac{10,000(1-z^{-1})}{17,265 + 2785 z^{-1}}$$

$$= \frac{10,000(1-z^{-1})}{17,265\left(1+\frac{2785}{17265} z^{-1}\right)}$$

$$\boxed{H(z) = \frac{0.579(1-z^{-1})}{1+0.1584 z^{-1}}}$$

Design a Second order Digital Low pass Butterworth Filter with a cutoff frequency 1 KHZ and Sampling frequency is 10 KHZ using Bilinear Transformation.

Soln: Given: Second order Digital LPF

$$\boxed{N=2}$$

Cut off frequency $f_c = 1\text{ KHZ} \Rightarrow 1\times10^3$

Sampling frequency $F_{SF} = 10\text{ KHZ} \Rightarrow 10\times10^3$

Discrete time frequency $\omega_c = 2\pi f_c$

$$f_c = \frac{F_c}{F_{SF}} = \frac{1000}{10000} \Rightarrow \boxed{f_c = 0.1}$$

$$W_c = 2\pi f_c$$

$$= 2\pi \times 0.1 \quad \boxed{W_c = 0.2\pi}$$

Step 1: order of the Filter N.

Given: $\boxed{N=2}$

Step 2: To Find H(S)

Second order Transfer Function $\boxed{H(s) = \frac{1}{s^2+\sqrt{2}s+1}}$ Refer Butterworths polynomial table

Step 3: To find cut off frequency.

Using Bilinear Transformation

$$\Omega c = \frac{2}{T}\tan\left(\frac{\omega c}{2}\right) \qquad T = \frac{1}{f_s} = \frac{1}{10^3\times 10}$$

$$= \frac{2}{10^{-4}}\tan\left(\frac{0.2\pi}{2}\right) \qquad T = 10^{-4} = \frac{1}{10000}$$

$$\boxed{\Omega c = 0.649\times 10^4 \text{rad/sec}}$$

Step 4: To find $H_a(s)$ for value of Ωc

$$H_a(s) = H(s)\Big|_{s\to\frac{s}{\Omega c}}$$

$$= \frac{1}{s^2+\sqrt{2}s+1}\Bigg|_{s\to\frac{s}{0.649}}$$

$$= \frac{1}{\left(\frac{s}{0.649\times10^4}\right)^2+\sqrt{2}\left(\frac{s}{0.649\times10^4}\right)+1}$$

$$= \frac{1}{\frac{s^2}{10^8(0.649)^2}+\frac{\sqrt{2}s+0.649\times10^4}{0.649\times10^4}}$$

$$= \frac{(0.649)^3\times10^{12}}{10^4\times0.649s^2+(0.649)^2\times\sqrt{2}\overset{\times10^8}{s}+(0.649)^3\times10^{12}}$$

$$= \frac{(0.649)^{\cancel{3}}\times10^{12}}{\cancel{0.649}\left[10^4s^2+0.649\sqrt{2}\overset{\times10^8}{s}+(0.649)^2\times10^{12}\right]}$$

$$\boxed{H_a(s) = \frac{(0.649)^2 \times 10^{12}}{10^4 \times s^2 + 0.917 \overset{\times 10^8}{s} + (0.649)^2 \times 10^{12}}}$$

Step 5: To Find H(z)

Using Bilinear Transformation,

$$H(z) = H_a(s)\Big|_{s \to \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

$$= \frac{(0.649)^2 \times 10^{12}}{10^4 s^2 + 0.917 \overset{\times 10^8}{s} + (0.649)^2 \times 10^{12}}\Bigg|_{s \to \frac{2}{10^{-4}}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

$$= \frac{(0.649)^2 \times 10^{12}}{10^4\left[\frac{2}{10^{-4}}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right] + 0.917 \times 10^8 \left[\frac{2}{10^{-4}}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right] + (0.649)^2 \times 10^{12}}$$

$$= \frac{(0.649)^2 \times 10^{12}}{10^4\left[\frac{4(1-z^{-1})^2}{10^{-8}(1+z^{-1})^2}\right] + \frac{1.835 \times 10^8 (1-z^{-1})}{10^{-4}(1+z^{-1})} + (0.649)^2 \times 10^{12}]}$$

$$= \frac{10^{12} \times 0.421(1+z^{-1})^2}{10^{12} \times 4(1-z^{-1})^2 + 1.835 \times 10^{12}(1-z^{-1})(1+z^{-1}) + 0.421 \times 10^{12}(1+z^{-1})^2}$$

$$= \frac{\cancel{10^{12}} \times 0.421(1+z^{-1})^2}{\cancel{10^{12}}\left[4(1-2z^{-1}+z^{-2}) + 1.835(1+\cancel{z^{-1}}-\cancel{z^{-1}}-z^{-2}) + 0.421(1+2z^{-1}+z^{-2})\right]}$$

$$= \frac{0.421(1+z^{-1})^2}{4-8z^{-1}+4z^{-2}+1.835-1.835z^{-2}+0.421+0842z^{-1} + 0.421z^{-2}}$$

$$\boxed{H(z) = \frac{0.421(1+z^{-1})^2}{6.256-7.158z^{-1}+2.586z^{-2}}}$$

Design a Digital Butterworth Filter Statisfying the Constraints

$0.707 \leq |H(e^{j\omega})| \leq 1$, For $0 \leq \omega \leq \pi/2$

$|H(e^{j\omega})| \leq 0.2$, for $\frac{3\pi}{4} \leq \omega \leq \pi$ with $T = 1sec$

Using Bilinear Transformation.

Soln: Given: $\omega_p = \frac{\pi}{2}$, $\omega_s = \frac{3\pi}{4}$

$$\frac{1}{\sqrt{1+\varepsilon^2}} = 0.707$$

$$\frac{1}{0.707} = \sqrt{1+\varepsilon^2}$$

Squaring on both sides

$$\left(\frac{1}{0.707}\right)^2 = \left(\sqrt{1+\varepsilon^2}\right)^2$$

$$\frac{1}{0.4998} = 1+\varepsilon^2$$

$$2.00080-1 = \varepsilon^2$$

$$\sqrt{1.00081} = \varepsilon$$

$$\varepsilon = 1.0008$$

$$\boxed{\varepsilon = 1}$$

$$\frac{1}{\sqrt{1+\lambda^2}} = 0.2$$

$$\frac{1}{0.2} = \sqrt{1+\lambda^2}$$

Squaring on both sides.

$$\left(\frac{1}{0.2}\right)^2 = \left(\sqrt{1+\lambda^2}\right)^2$$

$$\frac{1}{0.04} = 1+\lambda^2$$

$$25-1 = \lambda^2$$

$$\sqrt{24} = \lambda^2$$

$$\boxed{\lambda = 4.8989}$$

To Find prewarping Analog Frequency (Bilinear Transform)

$$\Omega s = \frac{2}{T} \tan \frac{Ws}{2} \qquad \Omega p = \frac{2}{T} \tan \frac{Wp}{2}$$

$$= \frac{2}{1} \tan \frac{3\pi/4}{2} \qquad = \frac{2}{1} \tan \frac{\pi/2}{2}$$

$$\boxed{\Omega s = 4.8284 \text{ rad/sec}} \qquad \boxed{\Omega p = 2 \text{ rad/sec}}$$

Step 1: To Find the order of the Filter:

$$N \geq \frac{\log(\lambda/\varepsilon)}{\log(\Omega s/\Omega p)} \Rightarrow \geq \frac{\log\left(\frac{4.898}{1}\right)}{\log\left(\frac{4.8284}{2}\right)}$$

$$\geq \frac{0.69001}{0.38277} \Rightarrow \geq 1.8026$$

$$\boxed{N = 2}$$

Step 2: To Find H(s)

For N=2, Refer Butterworth polynomial Table

$$\boxed{H(s) = \frac{1}{s^2+\sqrt{2}s+1}}$$

Step 3: To Find the cut off frequency Ωc

$$\Omega c = \frac{\Omega p}{(\varepsilon)^{1/N}} \Rightarrow \frac{2}{(1)^{1/2}}$$

$$\boxed{\Omega c = 2 \text{ rad/sec}}$$

Step 4: To Find Ha(s) For value of Ωc

$$Ha(s) = H(s) \Big|_{s \to \frac{s}{\Omega c}}$$

$$= \frac{1}{s^2+\sqrt{2}s+1} \Big|_{s \to \frac{s}{\Omega c}}$$

$$= \frac{1}{\left(\frac{s}{2}\right)^2 + \sqrt{2}\left(\frac{s}{2}\right)+1}$$

$$= \frac{1}{\frac{s^2}{4} + \frac{\sqrt{2}s+2}{2}} \Rightarrow \frac{8}{2s^2+4\times\sqrt{2}\,s+8}$$

$$\Rightarrow \frac{\cancel{8}}{\cancel{2}(s^2+2.828s+4)} \Rightarrow \boxed{H(s)=\frac{4}{s^2+2.828s+4}}$$

Step 5: To Find H(z) Using Bilinear Transformation T=1sec

$$H(z) = H(s)\Big|_{s \to \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

$$= \frac{4}{\left(2\left[\frac{1-z^{-1}}{1+z^{-1}}\right]\right)^2 + 2.828\left(2\left[\frac{1-z^{-1}}{1+z^{-1}}\right]\right)+4}$$

$$= \frac{4}{\frac{4(1-z^{-1})^2}{(1+z^{-1})^2} + \frac{2.828\times2(1-z^{-1})+4(1+z^{-1})}{(1+z^{-1})}}$$

$$= \frac{4(1+z^{-1})^3}{4(1-z^{-1})^2(1+z^{-1}) + (1+z^{-1})^2\left[5.656(1-z^{-1})+4(1+z^{-1})\right]}$$

$$= \frac{4(1+z^{-1})^{\cancel{3}}}{\cancel{(1-z^{-1})}\left[4(1-z^{-1})^2+(1+z^{-1})\left[5.656(1-z^{-1})+4(1+z^{-1})\right]\right]}$$

$$= \frac{4(1+z^{-1})^2}{4\left[1-2z^{-1}+z^{-2}\right]+5.656\left[1-z^{-1}+z^{-1}-z^{-2}\right]+4\left[1+2z^{-1}+z^{-2}\right]}$$

$$= \frac{4(1+z^{-1})^2}{4-8z^{-1}+4z^{-2}+5.656-5656z^{-2}+4+8z^{-1}+4z^{-2}}$$

$$\boxed{H(z) = \frac{4(1+z^{-1})^2}{13.656-2.344z^{-2}}}$$

Design a Butterworth digital Filter Using the Bilinear Transformation. the specifications of the desired low-pass filter are,

$$0.9 \le |H(\omega)| \le 1 ; \quad 0 \le \omega \le \frac{\pi}{2}$$

$$|H(\omega)| \le 0.2 ; \quad \frac{3\pi}{4} \le \omega \le \pi$$

With $T = 1$ sec

Soln: Given: $\omega p = \pi/2$, $\omega s = \frac{3\pi}{4}$

To Find $\Omega s, \Omega p$ (Analog Prewarping frequency)

$$\Omega s = \frac{2}{T} \tan\left(\frac{\omega s}{2}\right)$$

$$= \frac{2}{1} \tan\left(\frac{3\pi/4}{2}\right) \Rightarrow 2\tan\left(\frac{3\pi}{8}\right)$$

$$\boxed{\Omega s = 4.828}$$

$$\Omega p = \frac{2}{T} \tan\left(\frac{\omega p}{2}\right)$$

$$= \frac{2}{1} \tan\left(\frac{\pi/2}{2}\right) \Rightarrow 2\tan\left(\frac{\pi}{4}\right)$$

$$\boxed{\Omega p = 2}$$

To Find λ, ε:

$$\frac{1}{\sqrt{1+\varepsilon^2}} = 0.9$$

$$\frac{1}{0.9} = \sqrt{1+\varepsilon^2}$$

Squaring on both sides

$$\left(\frac{1}{0.9}\right)^2 = \left(\sqrt{1+\varepsilon^2}\right)^2$$

$$\frac{1}{\sqrt{1+\lambda^2}} = 0.2$$

$$\frac{1}{0.2} = \sqrt{1+\lambda^2}$$

Squaring on both sides

$$\left(\frac{1}{0.2}\right)^2 = \left(\sqrt{1+\lambda^2}\right)^2$$

$$\frac{1}{0.81} = 1+\varepsilon^2$$

$$1.2345-1 = \varepsilon^2$$

$$\sqrt{0.2345} = \varepsilon$$

$$\boxed{\varepsilon = 0.4843}$$

$$\frac{1}{0.04} = 1+\lambda^2$$

$$25-1 = \lambda^2$$

$$\sqrt{24} = \lambda$$

$$\boxed{\lambda = 4.8989}$$

Step 1: order of the Filter N

$$N \geq \frac{\log(\lambda/\varepsilon)}{\log(\Omega_s/\Omega_p)} \Rightarrow N \geq \frac{\log\left(\frac{4.8989}{0.4842}\right)}{\log\left(\frac{4.828}{2}\right)}$$

$$\geq \frac{1.0050}{0.3827} \Rightarrow \geq 2.6260$$

$$\boxed{N = 3}$$

Step 2: To Find H(s)

For $N=3$, 3rd order Transfer function

$$\boxed{H(s) = \frac{1}{(s+1)(s^2+s+1)}}$$

Refer Butterworth Polynomial table.

Step 3: To Find cut-off frequency.

$$\Omega_c = \frac{\Omega_p}{(\varepsilon)^{1/N}} \Rightarrow \frac{2}{(0.4843)^{1/3}} \Rightarrow \frac{2}{0.7852}$$

$$\boxed{\Omega_c = 2.5467}$$

Step 4: To Find Ha(s) For Value of Ω_c

$$H_a(s) = H(s)\Big|_{s \to \frac{s}{\Omega_c}}$$

$$= \frac{1}{(s+1)(s^2+s+1)}\Bigg|_{s \to \frac{s}{2.5467}}$$

$$= \frac{1}{\left(\frac{s}{2.5467}+1\right)\left(\left(\frac{s}{2.5467}\right)^2+\left(\frac{s}{2.5467}\right)+1\right)}$$

$$= \frac{1}{\left(\frac{s+2.5467}{2.5467}\right)\left[\frac{s^2}{(2.5467)^2} + \frac{s+2.5467}{2.5467}\right]}$$

$$= \left[\frac{1}{\frac{s+2.5467}{2.5467}}\right]\left[\frac{1}{\frac{s^2(2.5467) + (2.5467)^2 s + (2.5467)^3}{(2.5467)^3}}\right]$$

$$= \left[\frac{2.5467}{s+2.5467}\right]\left[\frac{(2.5467)^{\cancel{3}}}{\cancel{2.5467}\left[s^2 + 2.5467\, s + (2.5467)^2\right]}\right]$$

$$\boxed{H_a(s) = \left[\frac{2.5467}{s+2.5467}\right]\left[\frac{(2.5467)^2}{s^2 + (2.5467)s + (2.5467)^2}\right]}$$

Step 5: To Find H(z) Using Bilinear Transformation T = 1 sec

$$H(z) = H(s)\Big|_{s \to \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

$$= \left[\frac{2.5467}{s+2.5467}\right]\left[\frac{(2.5467)^2}{s^2 + (2.5467)s + (2.5467)^2}\right]\Bigg|_{s \to \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

$$= \left[\frac{2.5467}{\frac{2}{1}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 2.5467}\right]\left[\frac{(2.5467)^2}{\left[\frac{2}{1}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]^2 + (2.5467)\left[\frac{2}{1}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right] + (2.5467)^2}\right]$$

$$= \left[\frac{2.5467}{2(1-z^{-1}) + 2.5467(1+z^{-1})}\right]\left[\frac{(2.5467)^2}{\frac{4(1-z^{-1})^2}{(1+z^{-1})^2} + \left(\frac{5.0934(1-z^{-1}) + (2.5467)^2(1+z^{-1})}{(1+z^{-1})}\right)}\right]$$

$$= \left[\frac{2.5467}{2 - 2z^{-1} + 2.5467 + 2.5467z^{-1}}\right]\left[\frac{(2.5467)^2}{\frac{4(1-z^{-1})^2(1+z^{-1}) + 5.0934(1-z^{-1})(1+z^{-1})^2 + (2.5467)^2(1+z^{-1})^3}{(1+z^{-1})3}}\right]$$

$$\left[\frac{2.5467}{4.5467+0.5467z^{-1}}\right]\left[\frac{(1+z^{-1})^3\,(2.5467)^2}{4(1+z^{-2}-2z^{-1})(1+z^{-1})+5.0934-5.0934z^{-1}(1+z^{-1})^2 + (2.5467)^2(1+z^{-1})^3}\right]$$

$$=\left[\frac{2.5467}{(4.5467+0.5467z^{-1})}\right]\left[\frac{(1+z^{-1})^{\cancel{3}}(2.5467)^2}{(1+\cancel{z^{-1}})\left[4+4z^{-2}-2z^{-1}+(5.0934-5.0934z^{-1})(1+z^{-1}) + 2.5467(1+z^{-2}+2z^{-1})\right]}\right]$$

$$=\frac{(2.5467)^3(1+z^{-1})^2}{(4.5467+0.5467z^{-1})\left(4-4z^{-2}-2z^{-1}+5.0934+\cancel{5.0934z^{-1}}-\cancel{5.0934z^{-1}} -5.0934z^{-2}+2.5467+2.5467z^{-2}+5.0934z^{-1}\right]}$$

$$=\frac{(2.5467)^3(1+z^{-1})^2}{(4.5467+0.5467z^{-1})(11.6401+3.0934z^{-1}-6.5467z^{-2})}$$

$$=\frac{(2.5467)^3(1+z^{-1})^2}{(52.9240+14.0647z^{-1}-29.7658z^{-2}+6.3636z^{-1} +1.6911z^{-2}-3.5790z^{-3})}$$

$$=\frac{(2.5467)^3(1+z^{-1})^2}{52.9240+20.4283z^{-1}-28.0747z^{-2}-3.5790z^{-3}}$$

$$=\frac{16.5170(1+z^{-1})^2}{52.9240\left[1+0.3859z^{-1}-0.53047z^{-2}-0.0676z^{-3}\right]}$$

$$\boxed{H(z)=\frac{0.3120(1+z^{-1})^2}{1+0.3859z^{-1}-0.53047z^{-2}-0.067z^{-3}}}$$

Design a Lowpass Butterworth Filter to satisfy passband Cutoff $= 0.2\pi$, Stopband cutoff $= 0.8\pi$, Passband ripple $= 7dB$ Stopband ripple $= 16dB$, $T = 1 sec$. Using Impulse Invarient Method

Soln: Given: Passband cutoff freq $\omega_p = 0.2\pi$

Stop band cut off freq $\omega_s = 0.3\pi$

Pass band ripple $\alpha_p = 7dB$

Stop band ripple $\alpha_s = 16dB$

Step 1: To Find the order of the Filter N.

$$N \geq \frac{\log\sqrt{\frac{(10^{0.1\alpha_s})-1}{(10^{0.1\alpha_p})-1}}}{\log(\Omega_s/\Omega_p)}$$

To Find Ω_s & Ω_p

Analog frequency for Impulse Invarient Method

$$\Omega_s = \frac{W_s}{T} \qquad \Omega_p = \frac{W_p}{T}$$

$$\Omega_s = \frac{0.3\pi}{1} \qquad \Omega_p = \frac{0.2\pi}{1}$$

$$\boxed{\Omega_s = 0.3\pi} \qquad \boxed{\Omega_p = 0.2\pi}$$

$$N \geq \frac{\log\sqrt{\frac{(10^{0.1\times16})-1}{(10^{0.1\times7})-1}}}{\log(0.3\pi/0.2\pi)} \Rightarrow N \geq \frac{\log\sqrt{\frac{38.8107}{4.0118}}}{\log(1.5)}$$

$$N \geq \frac{0.4928}{0.1761} \Rightarrow N \geq 2.79$$

$$\boxed{N = 3}$$

Step 2: To Find Transfer Function H(s) for N = 3

$$\boxed{H(s) = \frac{1}{(s+1)(s^2+s+1)}}$$

Refer Butterworth Polynomial table

Step 3: Calculate the Value of cutoff frequency Ωc

$$\Omega c = \frac{\Omega p}{\left(10^{0.1\alpha p}-1\right)^{\frac{1}{2N}}} \Rightarrow \frac{0.2\pi}{\left(10^{0.1\times 7}-1\right)^{1/2\times 3}}$$

$$= \frac{0.2\pi}{1.2605} \Rightarrow \boxed{\Omega c = 0.5 \text{ rad/sec}}$$

Step 4: Find the Transfer Function $H_a(s)$ for the Value of Ωc

$$H_a(s) = H(s)\Big|_{s \to \frac{s}{\Omega c}}$$

$$= \frac{1}{(s+1)(s^2+s+1)}\Bigg|_{s \to \frac{s}{0.5}}$$

$$= \frac{1}{\left(\frac{s}{0.5}+1\right)\left(\left(\frac{s}{0.5}\right)^2+\left(\frac{s}{0.5}\right)+1\right)} \Rightarrow \frac{1}{\left(\frac{s+0.5}{0.5}\right)\left(\frac{s^2}{(0.5)^2}+\frac{s+0.5}{0.5}\right)}$$

$$= \frac{1}{\left(\frac{s+0.5}{0.5}\right)\left(\frac{s^2 0.5+(0.5)^2 s+(0.5)^3}{(0.5)^3}\right)} \Rightarrow \frac{(0.5)^4}{(s+0.5)\left[0.5(s^2+0.5s+0.25)\right]}$$

$$= \boxed{H_a(s) = \frac{(0.125)}{(s+0.5)(s^2+0.5s+0.25)}}$$

Step 5: To Find H(z)

$$\frac{A}{s+0.5} + \frac{Bs+C}{s^2+0.5s+0.25}$$

$$A = \cancel{(s+0.5)} \times \frac{0.125}{\cancel{(s+0.5)}(s^2+.5 \cdots}$$

To find A, B, C Value

$$A = \frac{0.125}{s^2+0.5s+0.25}\Bigg|_{s=0.5}$$

$$A = \frac{0.125}{(-0.5)^2+(0.5\overset{-0.5}{\cancel{\times}})+0.25} \Rightarrow \frac{0.125}{0.\cancel{25}-0.\cancel{25}+0.25}$$

$$\boxed{A = 0.5}$$

$$A(s^2+0.5s+0.25)+(Bs+C)(s+0.5) = 0.125$$

$$As^2+0.5sA+A0.25+Bs^2+B0.5s+Cs+0.5C = 0.125$$

$$(A+B)s^2+s(0.5A+0.5B+C)+(0.25A+0.5C) = 0.125$$

$A+B=0$; Putting $A=0.5$ $\quad 0.5+B=0$ $\boxed{B=-0.5}$

$0.5A+0.5B+C=0 \Rightarrow 0.5\times0.5+0.5\times-0.5+C=0$ $\boxed{C=0}$

$0.25A+0.5C=0.125$

Sub A, B, c value we get,

$$H_a(s) = \frac{0.5}{s+0.5} - \frac{0.5s+0}{\underset{a}{s^2}+\underset{b}{0.5}s+\underset{c}{0.25}}$$

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a} \Rightarrow \frac{-0.5\pm\sqrt{(0.5)^2-4\times1\times0.25}}{2\times1}$$

$$\Rightarrow \frac{-0.5\pm\sqrt{0.25-1}}{2\times1} \Rightarrow \frac{-0.5\pm\sqrt{-0.75}}{2} \Rightarrow \frac{-0.5\pm j\sqrt{0.5}}{2}$$

$$= -0.25\pm j0.433$$

$$s^2+0.5s+0.25 = (\overset{a}{s+0.25}-\overset{b}{j0.433})(s+0.25+j0.433)$$

$$= (s+0.25)^2+(0.433)^2$$

$(a+b)(a.b) = a^2+b^2$

Puttip that above eqn

$$H_a(s) = \frac{0.5}{s+0.5} - \frac{0.5s}{(s+0.25)^2+(0.433)^2}$$

Rearrange the above eqn

$$= \frac{0.5}{s+05} - 0.5\left[\frac{s+0.25-0.25}{(s+0.25)^2+(0.433)^2}\right]$$

$$= \frac{0.5}{s+0.5} - 0.5\left[\frac{s+0.25}{(s+0.25)^2+(0.433)^2} - \frac{0.25}{(s+0.25)^2+(0.433)^2}\right]$$

Rearrange second term.

$$= \frac{0.5}{s+0.5} - 0.5\left[\frac{s+0.25}{(s+0.25)^2+(0.433)^2} - \frac{0.25 \times \frac{0.433}{0.433}}{(s+0.25)^2+(0.433)^2}\right]$$

$$= \frac{0.5}{s+0.5} - 0.5\left[\frac{s+0.25}{(s+0.25)^2+(0.433)^2}\right] + 0.29\left[\frac{0.433}{(s+0.25)^2+(0.433)^2}\right]$$

Using Impulse Invarient Transformation.

$$\frac{1}{s-P_k} \longrightarrow \frac{1}{1-e^{-P_kT}z^{-1}}$$

$$\frac{s+a}{(s+a)^2+b^2} \rightarrow \frac{1-e^{-aT}(\cos bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}$$

$$\frac{b}{(s+a)^2+b^2} \rightarrow \frac{e^{-aT}(\sin bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}$$

Put $T=1$, $P_k=-0.5$, $a=0.25$, $b=0.433$

$$H(z) = \frac{0.5}{1-e^{-0.5}z^{-1}} - 0.5\left[\frac{1-e^{-0.25}\cos(0.433)z^{-1}}{1-2e^{-0.25}\cos(0.433)z^{-1}+e^{-2(0.25)}z^{-2}}\right]$$

$$+ 0.29\left[\frac{e^{-0.25}(\sin 0.433)z^{-1}}{1-2e^{-0.25}\cos(0.433)z^{-1}+e^{-2(0.25)}z^{-2}}\right]$$

$$= \frac{0.5}{1-0.6z^{-1}} - \frac{0.5(1-0.7z^{-1})}{1-1.41z^{-1}+0.6z^{-2}} + \frac{0.1z^{-1}}{1-1.41z^{-1}+0.6z^{-2}}$$

$$= \frac{0.5}{1-0.6z^{-1}} - \frac{0.5-0.35z^{-1}+0.1z^{-1} \quad 1.7\times10^{-3}}{1-1.41z^{-1}+0.6z^{-2}}$$

$$\boxed{H(z) = \frac{0.5}{1-0.6z^{-1}} - \frac{0.5-0.34z^{-1}}{1-1.41z^{-1}+0.6z^{-2}}}$$

Method to Convert Analog Filter Into Digital Filter

(i) Impulse Invarient Method

(ii) Bilinear Transformation Method

Problem:

Convert the Analog Filter with system Transfer Function $H(s) = \frac{2}{(s+1)(s+2)}$ into digital Filter by means of the Following method with $T = 1$ sec.

Impulse Invarient Method: $\boxed{H(s) = \frac{1}{s+a} \rightarrow H(z) = \frac{1}{1-e^{-aT}z^{-1}}}$

Given: $H(s) = \frac{2}{(s+1)(s+2)}$, $T = 1$ sec

Using Partial fraction we write

$$H(s) = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

$$A = \cancel{(s+1)} \times \frac{2}{\cancel{(s+1)}(s+2)}\Bigg|_{s=-1} \Rightarrow \frac{2}{s+2}\Bigg|_{s=-1}$$

$$= \frac{2}{-1+2} \quad \boxed{A = 2}$$

$$B = \cancel{(s+2)} \times \frac{2}{(s+1)\cancel{(s+2)}}\Bigg|_{s=-2} \Rightarrow \frac{2}{-2+1} \Rightarrow \boxed{B = -2}$$

$$H(s) = \frac{2}{(s+1)} - \frac{2}{(s+2)}$$

Using Impulse Invarient Method $H(z) = \frac{2}{1-e^{-T}z^{-1}} - \frac{2}{1-e^{-2T}z^{-1}}$, $T = 1$ sec

$$H(z) = \frac{2}{1-e^{-1}z^{-1}} - \frac{2}{1-e^{-2}z^{-1}} \Rightarrow \frac{2}{1-0.3678z^{-1}} - \frac{2}{1-0.135z^{-1}}$$

$$= \frac{2(1-0.1353z^{-1}) - 2(1-0.3678z^{-1})}{(1-0.3678z^{-1})(1-0.1353z^{-1})}$$

$$\boxed{H(z) = \frac{0.465z^{-1}}{1-0.503z^{-1}+0.0497z^{-2}}}$$

(ii) Bilinear Transformation:

$$H(z) = H(s)\Big|_{s \to \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

$$= \frac{2}{(s+1)(s+2)}\Big|_{s \to \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

$$= \frac{2}{\left\{\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]+1\right\}\left\{\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]+2\right\}} \quad \because T = 1\text{sec}$$

$$\frac{2}{\left\{\frac{2(1-z^{-1})+(1-z^{-1})}{(1+z^{-1})}\right\}\left\{\frac{2(1-z^{-1})+2(1+z^{-1})}{(1+z^{-1})}\right\}}$$

$$= \frac{2(1+z^{-1})^2}{(2-2z^{-1}+1+z^{-1})(2-\cancel{2z^{-1}}+2+\cancel{2z^{-1}})} \Rightarrow \frac{\cancel{2}(1+z^{-1})^2}{(3-z^{-1})\times \cancel{4}_{2}}$$

$$= \frac{(1+z^{-1})^2}{6-2z^{-1}} \Rightarrow \frac{(1+z^{-1})^2}{6(1-\frac{2}{6}z^{-1})}$$

$$\boxed{H(z) = \frac{0.166(1+z^{-1})^2}{(1-0.33z^{-1})}}$$

Problem:

Find h(n) Using Impulse Invarient method for the given $H(s) = \frac{10}{s^2+7s+10}$. Assume T = 1 sec

Soln: Given $H(s) = \frac{10}{s^2+7s+10} \Rightarrow \frac{10}{(s+2)(s+5)}$

Using partial fraction we can write

$$\begin{array}{r|l} 5 & 10 \\ \hline & 2 \end{array}$$

$$H(s) = \frac{A}{(s+2)} + \frac{B}{(s+5)}$$

To Find A, B value

$$A = (s+2) \times \frac{10}{(s+2)(s+5)} \Big|_{s=-2} \Rightarrow \frac{10}{-2+5} \quad \boxed{A = 3.33}$$

$$B = (s+5) \times \frac{10}{(s+2)(s+5)} \Big|_{s=-5} \Rightarrow \frac{10}{-5+2} \quad \boxed{B = -3.33}$$

$$H(s) = \frac{3.33}{(s+2)} - \frac{3.33}{(s+5)}$$

Using Impulse Invarient method

$$\because \frac{1}{s-P_K} = \frac{1}{1-e^{-P_K T} z^{-1}}$$

$$H(z) = \frac{3.33}{1-e^{-2}z^{-1}} - \frac{3.33}{1-e^{-5}z^{-1}}$$

$$\boxed{H(z) = 3.33\left[\frac{1}{1-0.135z^{-1}} - \frac{1}{1-0.0067z^{-1}}\right]}$$

Problem:

Convert the Analog filter into a digital Filter whose system function is $H(s) = \frac{s+0.2}{(s+0.2)^2+9}$ Use Impulse Invarient technique. assume T = 1 sec.

Soln: $$H_a(s) = \frac{s+a}{(s+a)^2+b^2} \xrightarrow[\text{Impulse Invarient}]{\text{Using}} H(z) = \frac{1-e^{-aT}\cos(bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1} + 2e^{-aT} + z^{-2}}$$

a = 0.2, b = 3

$$H(s) = \frac{s+\overset{a}{0.2}}{(s+0.2)^2+(\underset{b}{3})^2}$$

$$H(z) = \frac{1 - e^{-0.2}(\cos 3) z^{-1}}{1 - 2e^{-0.2}(\cos 3) z^{-1} + e^{-2\times 0.2} z^{-1}}$$

$$\boxed{H(z) = \frac{1 + 0.81 z^{-1}}{1 + 1.62 z^{-1} + 0.67 z^{-2}}}$$

Problem: An analog filter has the following system function Convert this filter into a digital filter using backward difference for the derivate $H(s) = \frac{s+0.1}{(s+0.1)^2 + 9}$

Soln: $H(s) = \frac{s+0.1}{(s+0.1)^2 + 3^2}$, Here, $\sigma = 0.1$, $\Omega = 3$, $\omega r = \pi/4$

$$\Omega = \frac{2}{T} \tan \frac{\omega}{2}$$

$$T = \frac{2}{\Omega} \tan \frac{\omega}{2} \Rightarrow \frac{2}{3} \tan \frac{\pi/4}{2} \qquad \boxed{T = 0.276 \text{ sec}}$$

Bilinear Transformation,

$$H(z) = \left.\frac{s+0.1}{(s+0.1)^2 + 3^2}\right| s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$$

$$\frac{\frac{2}{0.276}\left[\frac{1-z^{-1}}{1+z^{-1}}\right] + 0.1}{\left\{\frac{2}{0.276}\left[\frac{1-z^{-1}}{1+z^{-1}}\right] + 0.1\right\}^2 + 9} \Rightarrow \frac{\left[7.246(1-z^{-1}) + 0.1(1+z^{-1})\right] / 1+z^{-1}}{\left[\frac{7.246(1-z^{-1}) + 0.1(1+z^{-1})}{1+z^{-1}}\right]^2 + 9}$$

$$\frac{\left[7.246 - 7.246 z^{-1} + 0.1 + 0.1 z^{-1}\right] / (1+z^{-1})}{52.505(1 - 2z^{-1} + z^{-2}) + 0.01(1 + 2z^{-1} + z^{-2}) + 9(1+z^{-1})^2 / (1+z^{-1})^4}$$

$$\frac{7.346 - 7.146 z^{-1} \; (1+z^{-1})}{52.505 - 105.01 z^{-1} + 52.505 z^{-2} + 0.01 + 0.02 z^{-1} + 0.01 z^{-2} + 9 + 18 z^{-1} + 9 z^{-2}}$$

$$= \frac{7.346 + 7.346 z^{-1} - 7.146 z^{-1} - 7.146 z^{-2}}{61.515 - 86.99 z^{-1} + 61.515 z^{-2}}$$

$$= \frac{7.346 + 0.2 z^{-1} - 7.146 z^{-2}}{61.515 - 86.99 z^{-1} + 61.515 z^{-2}}$$

$$= \frac{7.346\left[1 + 0.0272 z^{-1} - 0.973 z^{-2}\right]}{61.515\left[1 - 1.4141 z^{-1} + z^{-2}\right]}$$

$$\boxed{H(z) = \frac{0.1194\left[1 + 0.0272 z^{-1} - 0.973 z^{-2}\right]}{\left(1 - 1.414 z^{-1} + z^{-2}\right]}}$$

Problem: Use the backward difference for the derivative to convert analog LPF with system function $H(s) = \frac{1}{s+2}$

Soln: Given: $H(s) = \frac{1}{s+2}$

$$H(z) = H(s)\Big|_{s \to \frac{1-z^{-1}}{T}} \Rightarrow \frac{1}{s+2}\Big|_{s \to \frac{1-z^{-1}}{T}}$$

$$= \frac{1}{\frac{1-z^{-1}}{T} + 2} \Rightarrow \frac{1}{\frac{(1-z^{-1}) + 2T}{T}}$$

$$\boxed{H(z) = \frac{T}{(1+2T) + z^{-1}}}$$

What are the Properties of Bilinear Transformation?

Bilinear Transformation $s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$

1. Right hand side of s-plane is mapped outside of the unit circle
2. Left hand side of s-plane is mapped inside of the unit circle
3. $j\Omega$ axis in s-plane is mapped on unit circle of the s plane

Prewarping:

In IIR design using bilinear transformation the conversion of the specified digital frequencies to analog frequencies called prewarping.

Its necessary to eliminate the effect of warping on amplitude response

$$\Omega = \frac{2}{T} \tan \cdot \frac{\omega}{2}$$

frequency Warping:

In Bilinear Transformation, the relation between Analog and digital frequencies is Non-linear. when s-plane is mapped into z-plane using Bilinear transformation, the non-linear relationship Introduces distortion in frequency axis is called frequency warping.

Advantage of Direct form-II realization when compared to direct form-I realization ?

(i) In direct form-II realization, number of memory location required is less than the direct form-I realization.

(ii) Same delay is used for input and output

Advantage of Casecade Realization ?

Quantization Error can be minimized if we realize an LTI system in casecade form.

DisAdvantage of Direct form-1

→ Its extremely sensitive to parameter quantization

→ when the order of the filter N is large a small change in the Filter coefficient due to parameter quantization results large change in the location of poles and zeros of the system.

DESIGNING STEPS FOR CHEBYSHEV FILTER!

Step 1: From the given Specification Find the oder of the Filter.

$$N \geq \frac{\cosh^{-1}\sqrt{\frac{\left(10^{0.1\alpha s}\right)-1}{\left(10^{0.1\alpha p}\right)-1}}}{\cosh^{-1}\left(\Omega s/\Omega p\right)} \quad (\text{or}) \quad N \geq \frac{\cosh^{-1}\left(\lambda/\varepsilon\right)}{\cosh^{-1}\left(\frac{\Omega s}{\Omega p}\right)}$$

Step 2: find the Major and Minor axis of the Ellipse a, b Using the Following Formula

$$a = \Omega p\left[\frac{\mu^{\left(\frac{1}{N}\right)} - \mu^{-\left(1/N\right)}}{2}\right], \quad b = \Omega p\left[\frac{\mu^{\left(\frac{1}{N}\right)} + \mu^{-\left(\frac{1}{N}\right)}}{2}\right]$$

Where, $\mu = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}}$, $\varepsilon = \sqrt{10^{0.1\alpha p} - 1}$

Step 3: Calculate the poles of Chebyshev Filter which Lie on the Elipse by Using the Formula.

$$S_K = a \cos\phi_K + j\, b \sin\phi_K, \quad K = 1, 2 \dots N$$

Where, $\phi_K = \frac{\pi}{2} + \left[\frac{(2K-1)}{2N}\right]\pi$

Step 4: Find the Determinator polynomial of the Transfer Function.

$$D(s) = (s-s_1)(s-s_2)(s-s_3)\cdots\cdots(s-s_K)$$

Step 5: Find Numerator polynomial N(s) of H(s). the numerator of the Transfer Function depends on the Value of N.

(a) For N = odd, Subsitude $S=0$ in the denominator and Find the Value.

(b) For N = Even, Subsititude $S=0$ in the denominators polynomial and divide the results by $\sqrt{1+\varepsilon^2}$

Step 6: Find the Analog Transfer Function

$$H(s) = \frac{N(s)}{D(s)}$$

Step 7: Find the Digital Transfer Function $H(z)$

(i) For Bilinear Transformation:

$$H(z) = H(s)\Big|_{s \rightarrow \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

(ii) For Impulse Invariant Method:

$$H(s) = \sum_{i=1}^{N} \frac{c}{s+p_i} \xrightarrow[\text{to}]{\text{Transformed}} H(z) = \sum_{i=1}^{N} \frac{c}{1-e^{-p_i T} z^{-1}}$$

Problem:

Design a chebyshev Filter Using For Following Specification, $0.8 \leq |H(e^{j\omega})| \leq 1$, $0 \leq \omega \leq 0.2\pi$

$|H(e^{j\omega})| \leq 0.2$, $0.6\pi \leq \omega \leq \pi$

Soln: Given: $\omega_p = 0.2\pi$, $\omega_s = 0.6\pi$, $\frac{1}{\sqrt{1+\varepsilon^2}} = 0.8$, $\frac{1}{\sqrt{1+\lambda^2}} = 0.2$

To Find λ, ε:

$$\frac{1}{\sqrt{1+\varepsilon^2}} = 0.8$$

Squaring on both sides

$$\frac{1}{\sqrt{1+\lambda^2}} = 0.2$$

Squaring on both sides

$$\frac{1}{1+\varepsilon^2} = (0.8)^2$$

$$\frac{1}{0.64} = 1+\varepsilon^2$$

$$1.5625-1 = \varepsilon^2$$

$$\sqrt{0.5625} = \varepsilon$$

$$\boxed{\varepsilon = 0.75}$$

$$\frac{1}{1+\lambda^2} = (0.2)^2$$

$$\frac{1}{0.04} = 1+\lambda^2$$

$$25-1 = \lambda^2$$

$$\sqrt{24} = \lambda$$

$$\boxed{\lambda = 4.898}$$

To Find Analog frequency $\Omega s, \Omega p$

$$\Omega s = \frac{2}{T}\tan\left[\frac{\omega s}{2}\right]$$

Assume T=1

$$\Omega s = \frac{2}{1}\tan\left[\frac{0.6\pi}{2}\right]$$

$$\boxed{\Omega s = 2.7527}$$

$$\Omega p = \frac{2}{T}\tan\left[\frac{\omega p}{2}\right]$$

Assume T=1

$$\Omega p = \frac{2}{T}\tan\left[\frac{0.2\pi}{2}\right]$$

$$\boxed{\Omega p = 0.6498}$$

Step 1: To Find the order of Filter N

$$N \geq \frac{\cosh^{-1}(\lambda/\varepsilon)}{\cosh^{-1}(\Omega s/\Omega p)} \Rightarrow \geq \frac{\cosh^{-1}\left[\frac{4.898}{0.75}\right]}{\cosh^{-1}\left[\frac{2.7527}{0.6498}\right]}$$

$$\geq \frac{2.5637}{2.1225}$$

$\left[\because \cosh^{-1} = \text{shift} + \text{hyp} + \cos\right.$

$$N \geq 1.208 \Rightarrow \boxed{N = 2}$$

Step 2: To Find Major and Minor axis of Ellipse:

$$\mu = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}}$$

$$= (0.75)^{-1} + \sqrt{1+(0.75)^{-2}}$$

$$\boxed{\mu = 3}$$

$$a = \Omega_p\left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right]$$

$$= 0.6498\left[\frac{(3)^{1/2} - (3)^{-1/2}}{2}\right]$$

$$= 0.6498\left[\frac{1.7320 - 0.5774}{2}\right]$$

$$= 0.6492 \times 0.57783$$

$$\boxed{a = 0.3752}$$

$$b = \Omega_p\left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right]$$

$$= 0.6498\left[\frac{(3)^{1/2} + (3)^{-1/2}}{2}\right]$$

$$= 0.6498\left[\frac{1.7320 + 0.5774}{2}\right]$$

$$= 0.6498 \times (2.8094)/2$$

$$= 0.6498 \times 1.1547$$

$$\boxed{b = 0.75}$$

Step 8: To Find the Poles of the chebychev Filter:

$$S_K = a\cos\phi_K + jb\sin\phi_K\ ,\quad K = 1, 2, 3 \cdots N$$

Where, $\phi_K = \frac{\pi}{2} + \left[\frac{2k-1}{2N}\right]\pi\ ,\quad K = 1, 2\quad [\because N = 2$

K=1,

$$\phi_1 = \frac{\pi}{2} + \left[\frac{2\times1-1}{2\times2}\right]\pi \Rightarrow \frac{\pi}{2} + \frac{\pi}{4} = \frac{4\pi+2\pi}{8} \Rightarrow \frac{\cancel{6}^{3}\pi}{\cancel{8}_{4}}$$

$$\boxed{\phi_1 = 3\pi/4}$$

K=2,

$$\phi_2 = \frac{\pi}{2} + \left[\frac{2\times2-1}{2\times2}\right]\pi \Rightarrow \frac{\pi}{2} + \left(\frac{3}{4}\right)\pi \Rightarrow \frac{4\pi+6\pi}{8} = \frac{\cancel{10}^{5}\pi}{\cancel{8}_{4}}$$

$$\boxed{\phi_2 = 5\pi/4}$$

To Find S_1, S_2

K=1

$$S_1 = a\cos\phi_1 + jb\sin\phi_1$$

$$= 0.3752\cos\left(\frac{3\pi}{4}\right) + j\,0.75\sin\left(\frac{3\pi}{4}\right)$$

$$= (0.3752\times -0.7071) + j(0.75\times 0.7071)$$

$$\boxed{S_1 = -0.2653 + j\,0.53}$$

$k=2,\ S_2 = a\cos\phi_2 + j\, b\sin\phi_2$

$= 0.8752\cos\left(\frac{5\pi}{4}\right) + j\,0.75\sin\left(\frac{5\pi}{4}\right)$

$= (0.8752 \times -0.7071) + j(0.75 \times -0.7071)$

$$\boxed{S_2 = -0.2653 - j\,0.531}$$

Step 4: To Find D(s)

$D(s) = (s-s_1)(s-s_2)$ for $N=2$

$D(s) = \left[s-(-0.2653+j0.53)\right]\left[s-(-0.2653-j0.53)\right]$

$= \left[\underset{a}{(s+0.2653)} - j\underset{b}{0.53}\right]\left[\underset{a}{(s+0.2653)} + j\underset{b}{0.53}\right]$

$= (s+0.2653)^2 + (0.53)^2$

$= s^2 + (0.2653)^2 + (2\times 0.2653\,s) + 0.2809$

$= s^2 + 0.0704 + 0.5306s + 0.2809$

$$\boxed{D(s) = s^2 + 0.5306\,s + 0.851}$$

Step 5: To Find N(s)

To find Numerator put $s=0$, and divide the value by $\sqrt{1+\varepsilon^2}$ for. even N $\quad \left[\because \text{for } N=2 \text{ Even}\right.$

$N(s) = \dfrac{0 + 0.5306\times 0 + 0.351}{\sqrt{1+\varepsilon^2}} \Rightarrow \dfrac{0.351}{\sqrt{1+(0.75)^2}}$

$= \dfrac{0.351}{1.25}$

$$\boxed{N(s) = 0.28}$$

Step 6: Find Analog Transfer function:

$$H(s) = \frac{N(s)}{D(s)}$$

$$\boxed{H(s) = \frac{0.28}{s^2 + 0.5306s + 0.351}}$$

Problem 2:

Design a digital chebyshev Filter to satisfy the constrains, $0.707 \leq |H(e^{j\omega})| \leq 1, \quad 0 \leq \omega \leq 0.2\pi$

$|H(e^{j\omega})| \leq 0.1, \quad 0.5\pi \leq \omega \leq \pi$

Using Bilinear Transformation and assuming $T = 1$ sec

Given: $\omega_p = 0.2\pi$, $W_s = 0.5\pi$, $\frac{1}{\sqrt{1+\varepsilon^2}} = 0.707$,

$\frac{1}{\sqrt{1+\lambda^2}} = 0.1$

To Find prewarping frequency [Bilinear Transformation]

Ω_s, Ω_p

$$\Omega_s = \frac{2}{T}\tan\left(\frac{\omega_s}{2}\right)$$

$$\Omega_s = \frac{2}{1}\tan\left(\frac{0.5\pi}{2}\right)$$

$$= 2 \times 1$$

$\boxed{\Omega_s = 2}$ rad/sec

$$\Omega_p = \frac{2}{T}\tan\left(\frac{\omega_p}{2}\right)$$

$$\Omega_p = \frac{2}{1}\tan\left(\frac{0.2\pi}{2}\right)$$

$$= 2 \times 0.3249$$

$\boxed{\Omega_p = 0.6498}$ rad/sec

To Find λ, ε

$$\frac{1}{\sqrt{1+\varepsilon^2}} = 0.707$$

Squaring on both sides

$$\left(1/\sqrt{1+\varepsilon^2}\right)^2 = (0.707)^2$$

$$\frac{1}{\sqrt{1+\lambda^2}} = 0.1$$

Squaring on both sides

$$\left(1/\sqrt{1+\lambda^2}\right)^2 = (0.1)^2$$

$$\frac{1}{(0.707)^2} = 1+\varepsilon^2$$

$$2-1 = \varepsilon^2$$

$$\sqrt{1} = \varepsilon$$

$$\boxed{\varepsilon = 1}$$

$$\frac{1}{(0.1)^2} = 1+\lambda^2$$

$$100-1 = \lambda^2$$

$$\sqrt{99} = \lambda$$

$$\boxed{\lambda = 9.95}$$

Step 1: To find the order of the Filter N

$$N \geq \frac{\cosh^{-1}(\lambda/\varepsilon)}{\cosh^{-1}(\Omega s/\Omega p)} \Rightarrow \geq \frac{\cosh^{-1}\left(\frac{9.95}{1}\right)}{\cosh^{-1}(2/0.6498)}$$

$$\geq \frac{2.9882}{1.7899} \Rightarrow \geq 1.669$$

order of the Filter $\boxed{N = 2}$

Step 2: To find Major and Minor axis of Ellipse

$$\mu = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}} \Rightarrow (1)^{-1} + \sqrt{1+(1)^{-2}}$$

$$\Rightarrow 1+\sqrt{2}$$

$$\boxed{\mu = 2.414}$$

$$a = \Omega p\left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right]$$

$$a = 0.6498\left[\frac{(2.414)^{1/2} - (2.414)^{-1/2}}{2}\right]$$

$$= 0.6498\left[\frac{1.5537 - 0.6436}{2}\right]$$

$$= 0.6498 \times 0.4551$$

$$\boxed{a = 0.2957}$$

$$b = \Omega p\left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right]$$

$$b = 0.6498\left[\frac{(2.414)^{1/2} + (2.414)^{-1/2}}{2}\right]$$

$$= 0.6498\left[\frac{1.5537 + 0.6436}{2}\right]$$

$$= 0.6498 \times 1.0987$$

$$\boxed{b = 0.7139}$$

Step 3: To find the poles of the chebyshev filter:

$$s_k = a\cos\phi_k + j\,b\sin\phi_k\,, \quad k = 1, 2, \cdots N$$

$$\phi_k = \frac{\pi}{2} + \left[\frac{2k-1}{2N}\right]\pi\,, \quad k = 1, 2, \;\because \text{For } N = 2$$

$$k=1, \quad \phi_1 = \frac{\pi}{2} + \left[\frac{2\times1-1}{2\times2}\right]\pi \Rightarrow \frac{\pi}{2} + \frac{\pi}{4} \Rightarrow \boxed{\phi_1 = \frac{3\pi}{4}}$$

$$k=2, \quad \phi_2 = \frac{\pi}{2} + \left[\frac{2\times2-1}{2\times2}\right]\pi \Rightarrow \frac{\pi}{2} + \frac{3\pi}{2} \Rightarrow \boxed{\phi_2 = 5\pi/4}$$

$$s_1 = a\cos\phi_1 + j\,b\sin\phi_1$$

$$= 0.2957\cos(3\pi/4) + j\,0.7139\sin(3\pi/4)$$

$$= (0.2957 \times -0.7071) + j(0.7139 \times 0.7071)$$

$$\boxed{s_1 = -0.2091 + j\,0.5048}$$

$$s_2 = a\cos\phi_2 + j\,b\sin\phi_2$$

$$= 0.2957\cos(5\pi/4) + j\,0.7139\sin(5\pi/4)$$

$$= (0.2957 \times -0.7071) + j(0.7139 \times -0.7071)$$

$$\boxed{s_2 = -0.2091 - j\,0.5048}$$

Step 4: To find $D(s)$

$$D(s) = (s - s_1)(s - s_2)$$

$$= \left[s - (-0.2091 + j0.5048)\right]\left[s - (-0.2091 - j0.5048)\right]$$

$$= \left[(s + \underset{a}{0.2091}) - j\underset{b}{0.5048}\right]\left[(s + \underset{a}{0.2091}) + j\underset{b}{0.5048}\right]$$

$$= (s + 0.2091)^2 + (0.5048)^2$$

$$= s^2 + 0.0487 + 0.4182\,s + 0.2548$$

$$\boxed{D(s) = s^2 + 0.4182\,s + 0.2985}$$

Step 5: To find N(s)

For $N = 2$, Even, To Find Numerator put $s = 0$, and divide the value by $\sqrt{1+\varepsilon^2}$

$$N(s) = \frac{0 + 0.4182 \times 0 + 0.2985}{\sqrt{1+\varepsilon^2}} \Rightarrow \frac{0.2985}{\sqrt{1+1^2}}$$

$$\boxed{N(s) = 0.2111}$$

Step 6: To find Transfer Function H(s)

$$H(s) = \frac{N(s)}{D(s)}$$

$$\boxed{H(s) = \frac{0.2111}{s^2 + 0.4182\,s + 0.2985}}$$

Step 7: Find the Digital Transfer Function H(z)

for Bilinear Transformation!

$$H(z) = H(s)\Big|_{s \to \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

$$= \frac{0.2111}{s^2 + 0.4182\,s + 0.2985}\Bigg|_{s \to \frac{2}{1}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

$$= \frac{0.2111}{\left(2\left[\frac{1-z^{-1}}{1+z^{-1}}\right]\right)^2 + 0.4182\left[\frac{2(1-z^{-1})}{(1+z^{-1})}\right] + 0.2985}$$

$$= \frac{0.2111}{\frac{4(1-z^{-1})^2}{(1+z^{-1})^2} + \frac{0.8364(1-z^{-1})}{(1+z^{-1})} + 0.2985}$$

$$= \frac{0.2111}{4\frac{(1-z^{-1})^2}{(1+z^{-1})^2} + \frac{0.8364(1-z^{-1}) + 0.2985(1+z^{-1})}{(1+z^{-1})}}$$

$$= \frac{0.2111(1+z^{-1})^3}{4(1-z^{-1})^2(1+z^{-1}) + 0.8364(1-z^{-1})(1+z^{-1})^2 + 0.2985(1+z^{-1})^3}$$

$$= \frac{0.2111(1+z^{-1})^{\cancel{3}}}{\cancel{(1+z^{-1})}\left[4(1+z^{-2}-2z^{-1}) + 0.8364(1+\cancel{z^{-1}}-\cancel{z^{-1}}-z^{-2}) + 0.2985(1+z^{-2}+2z^{-1})\right]}$$

$$= \frac{0.2111(1+z^{-1})^2}{4+4z^{-2}-8z^{-1}+0.8364-0.8364z^{-2}+0.2985+0.2985z^{-2}+0.597z^{-1}}$$

$$= \frac{0.2111(1+z^{-1})^2}{5.1349+3.4621z^{-2}-7.403z^{-1}}$$

$$= \frac{0.2111(1+z^{-1})^2}{5.1349\left[1+0.6742z^{-2}-1.4417z^{-1}\right]}$$

$$\boxed{H(z) = \frac{0.0411(1+z^{-1})^2}{1-1.4417z^{-1}+0.6742z^{-2}}}$$

Problem 3:

Give the specifications $\alpha_p = 3dB$, $\alpha_s = 16dB$, $f_p = 1$ KHz, $f_s = 2$ KHz. Determine the order of the Filter Using chebyshev approximation. Find H(s).

Given: $\alpha_p = 3dB$, $\alpha_s = 16dB$

$f_p = 1$ KHz, $f_s = 2$ KHz

$$\Omega p = 2\pi f_p \Rightarrow 2\pi \times 1 \times 10^3 \Rightarrow \boxed{\Omega p = 2000\pi} \text{ rad/sec}$$

$$\Omega s = 2\pi fs \Rightarrow 2\pi \times 2 \times 10^3 \Rightarrow \boxed{\Omega s = 4000\pi} \text{ rad/sec}$$

Step 1: To find the order of the Filter N

$$N \geq \frac{\cosh^{-1}\sqrt{\dfrac{(10^{0.1\alpha s})-1}{(10^{0.1\alpha p})-1}}}{\cosh^{-1}(\Omega s/\Omega p)} \Rightarrow \geq \frac{\cosh^{-1}\sqrt{\dfrac{(10^{0.1\times 16})-1}{(10^{0.1\times 3})-1}}}{\cosh^{-1}\left(\dfrac{4000\pi}{2000\pi}\right)}$$

$$\geq \frac{\cosh^{-1}\sqrt{\left(\dfrac{38.81}{0.995}\right)}}{\cosh^{-1}(2)} \Rightarrow \geq \frac{2.5185}{1.3169}$$

$$N \geq 1.912 \Rightarrow \boxed{N = 2}$$

Step 2: Find the Major and Minor axis of the Ellipse a, b using the following formula.

$$\varepsilon = \sqrt{(10^{0.1\alpha p})-1} \Rightarrow \sqrt{(10^{0.1\times 3})-1} \Rightarrow \boxed{\varepsilon = 0.998}$$

$$\mu = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}} \Rightarrow (0.998)^{-1} + \sqrt{1+(0.998)^{-2}}$$

$$= 1.002 + 1.4156 \quad \boxed{\mu = 2.4176}$$

$$a = \Omega p\left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right]$$

$$a = 2000\pi\left[\frac{(2.4176)^{1/2} - (2.4176)^{-1/2}}{2}\right]$$

$$= 2000\pi\left[\frac{1.5549 - 0.6431}{2}\right]$$

$$= 2000\pi \times 0.4559$$

$$\boxed{a = 911.8\pi}$$

$$b = \Omega p\left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right]$$

$$b = 2000\pi\left[\frac{(2.4176)^{1/2} + (2.4176)^{-1/2}}{2}\right]$$

$$= 2000\pi\left[\frac{1.5549 + 0.6431}{2}\right]$$

$$= 2000\pi \times 1.099$$

$$\boxed{b = 2198\pi}$$

Step 3: To Find the Poles of the chebyshev Filter

$$S_K = a \cos \phi_K + j\, b \sin \phi_K , \quad K = 1, 2 \ldots N$$

Where $\phi_K = \frac{\pi}{2} + \left[\frac{2K-1}{2N}\right] \pi, \quad K = 1, 2 \; [\because \text{For } N = 2$

Where, $K=1 \quad \phi_1 = \frac{\pi}{2} + \left[\frac{2\times1-1}{2\times2}\right]\pi \Rightarrow \frac{\pi}{2} + \frac{\pi}{4} \quad \boxed{\phi_1 = \frac{3\pi}{4}}$

$K=2, \quad \phi_2 = \frac{\pi}{2} + \left[\frac{2\times2-1}{2\times2}\right]\pi \Rightarrow \frac{\pi}{2} + \frac{3\pi}{2} \quad \boxed{\phi_2 = \frac{5\pi}{4}}$

then, $S_1 = a \cos \phi_1 + j \sin \phi_1 \, b$

$$= 911.8\pi \cos\left(\frac{3\pi}{4}\right) + j \; 2198\pi \sin\left(3\pi/4\right)$$

$$= (911.8\pi \times -0.7071) + j(2198\pi \times 0.7071)$$

$$\boxed{S_1 = -644.73\pi + j\,1554.2\pi}$$

$S_2 = a \cos \phi_2 + j \sin \phi_2 \; b$

$$= 911.8\pi \cos(5\pi/4) + j \; 2198\pi \sin(5\pi/4)$$

$$= (911.8\pi \times -0.707) + j(2198\pi \times -0.7071)$$

$$\boxed{S_2 = -644.73\pi - j1554.2\pi}$$

Step 4: To Find D(s)

$$D(s) = (s - s_1)(s - s_2)$$

$$= \left[s - (-644.73\pi + j1554.2\pi)\right]\left[s - (-644.73\pi - j1554.2\pi)\right]$$

$$= \left[\underbrace{(s + 644.73\pi)}_{a} - j\underbrace{1554.2\pi}_{b}\right]\left[\underbrace{(s + 644.73\pi)}_{a} + j\underbrace{1554.2\pi}_{b}\right]$$

$$= (s + 644.73\pi)^2 + (1554.2\pi)^2$$

$$\boxed{D(s) = s^2 + (644.73\pi)^2 + 1289.5s + (1554.2\pi)^2}$$

Step5: To Find N(s)

- For N=2, Even, To find Numerator put $s=0$, and
- divide the value by $\sqrt{1+\varepsilon^2}$

$$N(s) = \frac{D(s)\big|_{s=0}}{\sqrt{1+\varepsilon^2}} = \frac{0+(644.78\pi)^2+0+(1554.2\pi)^2}{\underset{1.4128}{\sqrt{1+(0.998)^2}}}$$

$$\boxed{N(s) = (1415.61)^2\pi^2}$$

Step 6: To Find H(s)

$$H(s) = \frac{N(s)}{D(s)}$$

$$\boxed{H(s) = \frac{(1415.61)^2\pi^2}{s^2+(644.73\pi)^2+1289.5s+(1554.2\pi)^2}}$$

Problem: 1

Determine the direct Form realization of system Function H(z)

$1+3z^{-1}-4z^{-2}-4z^{-3}+6z^{-4}+5z^{-5}$

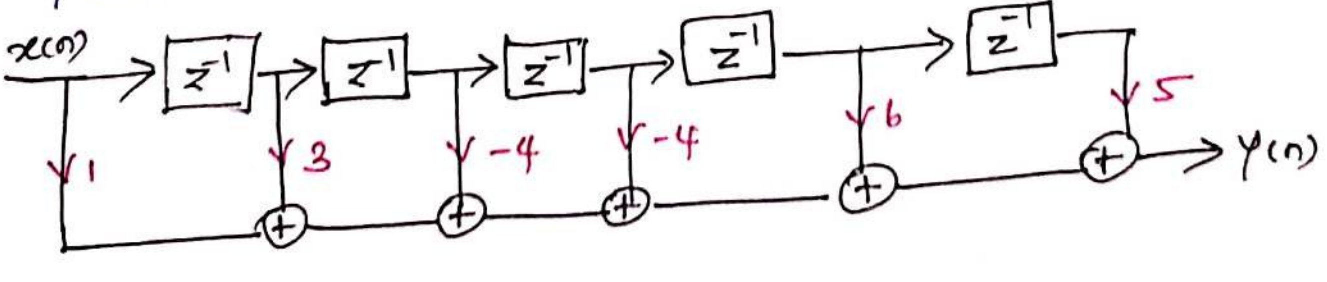
Soln:

$$\frac{Y(z)}{X(z)} = 1+3z^{-1}-4z^{-2}-4z^{-3}+6z^{-4}+5z^{-5}$$

$$Y(z) = X(z)+3z^{-1}X(z)\overset{-4z^{-2}X(z)}{\wedge}-4z^{-3}X(z)+6z^{-4}X(z)+5z^{-5}X(z)$$

Taking Inverse z Transform

$$y(n) = x(n)+3x(n-1)-4x(n-2)-4x(n-3)+6x(n-4)+5x(n-5)$$



Problem 2:

Obtain the direct form realization for the system function $H(z) = 1+\frac{1}{2}z^{-1}+\frac{1}{3}z^{-2}+\frac{1}{4}z^{-3}+\frac{1}{5}z^{-4}$

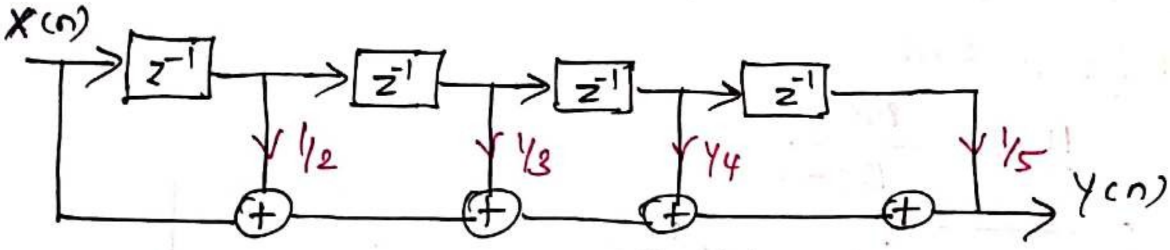
Soln:

$$\frac{Y(z)}{X(z)} = 1+\frac{1}{2}z^{-1}+\frac{1}{3}z^{-2}+\frac{1}{4}z^{-3}+\frac{1}{5}z^{-4}$$

$$Y(z) = X(z)+\frac{1}{2}z^{-1}X(z)+\frac{1}{3}z^{-2}X(z)+\frac{1}{4}z^{-3}X(z)+\frac{1}{5}z^{-4}X(z)$$

Taking Inverse z-transform we get

$$y(n) = x(n)+\frac{1}{2}x(n-1)+\frac{1}{3}x(n-2)+\frac{1}{4}x(n-3)+\frac{1}{5}x(n-4)$$



Design of FIR Filters - Symmetric and Anti-symmetric FIR Filters - design of Linear phase FIR filters Using Fourier Series Method - FIR filter design using windows [Rectangular, Hamming and Hanning window], frequency Sampling method, FIR filter structures - Linear phase structure, Direct Form Realizations.

Design of FIR Filters:

→ FIR Filter: → Impulse response h(n) is finite in duration.
→ output depends only on present and (or) past values.

Application of FIR Filter:

1. Data Transmission
2. speech processing
3. Correlation processing
4. Interpolation.

Properties of FIR Digital Filters:

(i) FIR Filters are Inherently stable.
(ii) FIR filter have Linear phase
(iii) FIR filters need Higher orders for Similar magnitude response Compared to IIR Filters.

Stability of FIR Filters (or) What is the reason that FIR filter is always stable ?

→ FIR Filters are Inherently stable. They are all zero filters. the poles of FIR filters are Located at origin in Z-plane.

This means FIR filter poles are always inside the unit circle. Hence FIR filters are always stable.

Output of FIR Filter $\boxed{y(n) = \sum_{k=0}^{N-1} b_k x(n-k)}$

$b_k \rightarrow$ Co-efficient of FIR Filter

They are finite and bounded. hence o/p y(n) will be bounded as long as x(n-k) is bounded.

Gibb's Phenomenon:

The desired Impulse response of FIR Filter is passed through the window of finite length to make the length of Impulse response finite. this generates oscillations (or) ringing near the band edge cut-off frequency of the Filter. this oscillatory behaviour is called Gibb's Phenomenon.

Gibbs phenomenon indicates the index of Imperfect filtering. effects are made to reduce the oscillatory behavior near W_c by using different types of windows.

Comparision of FIR and IIR Filters:

IIR Filters	FIR Filters
Unit Impulse response h(n) is infinite in duration	unit Impulse response h(n) is finite in duration.
Poles as well as zeros are present.	These are all zero filters
these filters use feedback from output. they are recursive filter	these filters do not use feedback. they are non-recursive.
these filters are to be designed for stability	these are Inherently stable filters

Advantage of FIR Filters:

(i) FIR Filters are inherently stable

(ii) FIR Filter can be designed for Linear phase

(iii) FIR Filter Can be realized on both recursive and non-recursive structure

DisAdvantage of FIR Filters:

(i) order of the filter is high. for same attenuation Characteristics Compared to IIR Filters.

(ii) Large storage requirement needed.

(iii) Powerful computation facility required for the Implementation

FIR Filter Design Methods:

(i) Fourier Series Method

(ii) Windowing Method

(iii) Frequency Sampling Method.

Steps to Design FIR Filter Using Windowing Techniques:

Step 1: obtain the desired Impulse response $h_d(n)$

From the given frequency response using the Formula

$$\boxed{h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{-j\omega n}\, d\omega} \quad (\text{or})$$

Use the $h(n)$ value From the table

Desired Filter Coefficient for Different types of Filter:

Low pass Filter:

$h_d(n)$ for Linear phase:	$h_d(n)$ for zero phase:
$h_d(n) = \dfrac{\sin\omega_c(n-\alpha)}{\pi(n-\alpha)}$ for $n \neq \alpha$	$h_d(n) = \dfrac{\sin\omega_c n}{\pi n}$ for $n \neq \alpha$
$h_d(n) = \dfrac{\omega_c}{\pi}$ for $n = \alpha \left(\frac{0}{0}\right)$	$h_d(n) = \dfrac{\omega_c}{\pi}$ for $n = \alpha \left(\frac{0}{0}\right)$

Highpass filter

Linear phase:

$$h_d(n) = \frac{\sin\pi(n-\alpha) - \sin\omega_c(n-\alpha)}{\pi(n-\alpha)}$$

for $n \neq \alpha$

$$h_d(n) = 1 - \frac{\omega_c}{\pi} \quad \text{for } n = \infty$$

(or) $\left[\frac{\pi - \omega_c}{\pi}\right]$

Zero phase:

$$h_d(n) = \frac{\sin\pi c - \sin\omega_c n}{\pi n} \quad \text{for } n \neq \alpha$$

$$h_d(n) = \frac{\pi - \omega_c}{\pi} \quad \text{for } n = \infty$$

Band pass filter:

Linear phase

$$h_d(n) = \frac{\sin\omega_{c_2}(n-\alpha) - \sin\omega_c(n-\alpha)}{\pi(n-\alpha)}$$

for $n \neq \alpha$

$$h_d(n) = \frac{\omega_{c_2} - \omega_{c_1}}{\pi}, \quad \text{for } n = \infty$$

Zero phase:

$$h_d(n) = \frac{\sin\omega_{c_2} n - \sin\omega_{c_1} n}{\pi n}$$

For $n \neq \alpha$

$$h_d(n) = \frac{\omega_{c_2} - \omega_{c_1}}{\pi} \quad \text{for } n = \infty$$

Band stop (or) Band Rejected Filter

Linear phase:

$$h_d(n) = \frac{\left[\sin\omega_{c_1}(n-\alpha) - \sin\omega_{c_2}(n-\alpha) + \sin\pi(n-\alpha)\right]}{\pi(n-\alpha)}$$

For $n \neq \alpha$

$$h_d(n) = \frac{\pi - \omega_{c_2} + \omega_{c_1}}{\pi} \quad \text{for } n = \infty$$

Zero phase:

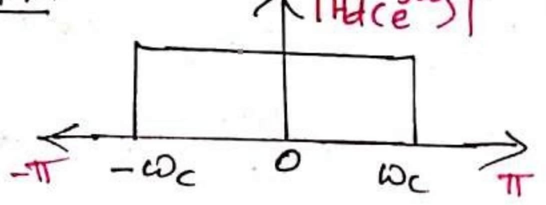
$$h_d(n) = \frac{\left[\sin\omega_{c_1} n - \sin\omega_{c_2} n + \sin\pi n\right]}{\pi n}$$

For $n \neq \alpha$

$$h_d(n) = \frac{\pi - \omega_{c_2} + \omega_{c_1}}{\pi} \quad \text{for } n = \infty$$

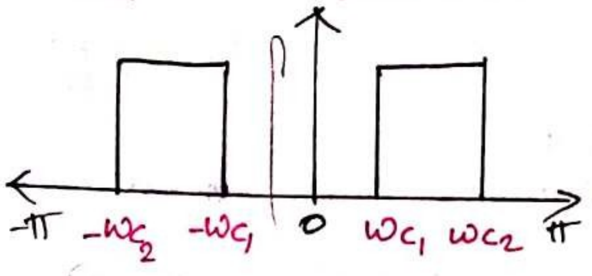
Types of Filter and and Magnitude response:

LPF:

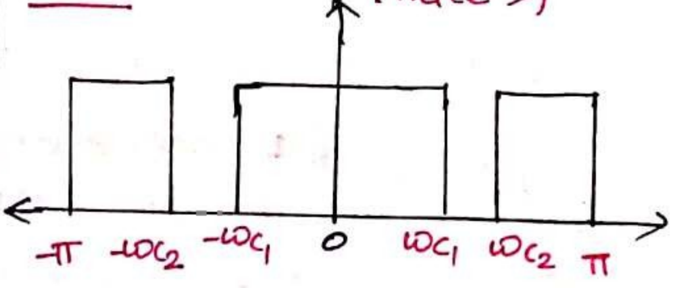


HPF

BPF



BRF



Step 2: To Find the window function $w(n)$

Types of window:

(i) Hamming window
(ii) Hanning window
(iii) Rectangular window

(i) Hamming window:

Causal Hamming window function for Linear phase filter

$$w_{Hm}(n) = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right) & \text{for } 0 \le n \le N-1 \\ 0 & \text{for otherwise} \end{cases}$$

Non causal Hamming window function for Zero phase filter

$$w_{Hm}(n) = \begin{cases} 0.54 + 0.46\cos\left(\frac{2\pi n}{N-1}\right) & \text{for } -\left(\frac{N-1}{2}\right) \le n \le \left(\frac{N-1}{2}\right) \\ 0 & \text{for otherwise} \end{cases}$$

(ii) Hanning window:

The causal Hanning window function for Linear phase filter

$$w_{Hn}(n) = \begin{cases} 0.5 - 0.5\cos\left(\frac{2\pi n}{N-1}\right), & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

Non causal Hanning window function for Zerophase filter

$$w_{Hn}(n) = \begin{cases} 0.5 + 0.5\cos\left(\frac{2\pi n}{N-1}\right), & -\left(\frac{N-1}{2}\right) \le n \le \left(\frac{N-1}{2}\right) \\ 0 & \text{otherwise} \end{cases}$$

(iii) Rectangular window:

Causal Rectangular window Function for Linear phase Filter

$$W_R(n) = \begin{cases} 1 & \text{for } 0 \le n \le N-1 \\ 0 & \text{for otherwise} \end{cases}$$

Non causal Rectangular window Function Zero phase Filter

$$W_R(n) = \begin{cases} 1 & \text{for } -\left(\frac{N-1}{2}\right) \le n \le \left(\frac{N-1}{2}\right) \\ 0 & \text{for otherwise} \end{cases}$$

Step 3: To Find h(n)

$$\boxed{h(n) = h_d(n)\, w(n)}$$

Step 4: Find the Transfer function H(z) by using the following formula.

Linear phase Filter:

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}$$

Zero phase Filter

$$H(z) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} h(n)\, z^{-n}$$

Step 5: To Find Frequency Response

Sub $z = e^{j\omega}$ in $H(z)$.

Symmetric / Asymmetric Condition:

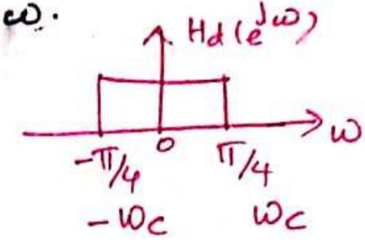
Filter type	Symmetric	Asymmetric
Linear phase filter $H_d(e^{j\omega}) = e^{-j\omega\alpha}$	$h(n) = h(N-1-n)$	$h(n) = -h(N-1-n)$
Zero phase Filter $H_d(e^{j\omega}) = 1$	$h(n) = h(-n)$	$h(n) = -h(-n)$

Problem 1

Design an Ideal filter with the frequency Response

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\overset{\alpha}{3}\omega} & \text{for } -\pi/4 \le \omega \le \pi/4 \\ 0 & \text{for otherwise} \end{cases}$$

Determine $h(n)$ for $N = 7$ use Hamming window.



Soln: Given: Types of filter : LPF

$N = 7$, $\omega_c = \pi/4$

$$\boxed{\alpha = \frac{N-1}{2}} \Rightarrow \alpha = \frac{7-1}{2} \quad \boxed{\alpha = 3}$$

Step I: To find $h_d(n)$

$\alpha = 3$, so given filter linear phase, symmetric condition.

$$\boxed{h_d(n) = \frac{\sin \omega_c(n-\alpha)}{\pi(n-\alpha)}}$$

$$h_d(n) = \frac{\sin \pi/4\,(n-3)}{\pi(n-3)}$$

The desired filter coefficient for symmetric condition:

$h(n) = h(N-1-n)$, with $N = 7$ | $N = 7$, $n = 0, 1, 2, 3, 4, 5, 6$

$n = 0$

$$h_d(0) = h_d(6) = \frac{\sin \pi/4(0-3)}{\pi(0-3)} \Rightarrow \frac{\sin(-\overset{\to 180^\circ}{3\pi/4})}{-\underset{\hookrightarrow 3.414}{3\pi}} = 0.075$$

$n = 1$:

$$h_d(1) = h_d(5) = \frac{\sin \pi/4(1-3)}{\pi(1-3)} \Rightarrow \frac{\sin(-\pi/2)}{-2\pi} = 0.159$$

$n = 2$:

$$h_d(2) = h(4) = \frac{\sin \pi/4(2-3)}{\pi(2-3)} \Rightarrow \frac{\sin(-\pi/4)}{\pi} = 0.225$$

$n = 3$:

$$h_d(3) = \frac{\sin \pi/4(3-3)}{\pi(3-3)} \Rightarrow \frac{\sin 0}{0} = \frac{0}{0} = \alpha \left(\text{undetermin}\right)$$

Use L Hospital Rule :

For $\left(\frac{0}{0}\right)$ condition the value of hd(3) is

$$\boxed{hd(3) = \frac{\omega_c}{\pi}} \Rightarrow \frac{\pi/4}{\pi} = \frac{1}{4} \quad \boxed{hd(3) = 0.25}$$

Step 2: To Find window sequence :

The causal Hamming window sequence.

$$\boxed{\omega_{Hm}(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right) : \quad 0 \le n \le N-1}$$

$$\omega_{Hm}(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{6}\right) ; \quad 0 \le n \le 6$$

$$\omega_{Hm}(0) = \omega_{Hm}(6) = 0.54 - 0.46\cos(0) = \boxed{0.08}$$

$$\omega_{Hm}(1) = \omega_{Hm}(5) = 0.54 - 0.46\cos\left(\frac{2\pi}{6}\right) = \boxed{0.31}$$

$$\omega_{Hm}(2) = \omega_{Hm}(4) = 0.54 - 0.46\cos\left(\frac{2\pi \times 2}{6}\right) = \boxed{0.77}$$

$$\omega_{Hm}(3) = 0.54 - 0.46\cos\left(\frac{2\pi \times 3}{6}\right) = \boxed{1}$$

Step 3: To Find h(n)

$$\boxed{h(n) = hd(n)\,\omega_{Hm}(n)}$$

$$h(0) = h(6) = hd(0)\,\omega_{Hm}(0) = 0.075 \times 0.08 = \boxed{0.006}$$

$$h(1) = h(5) = hd(1)\,\omega_{Hm}(1) = 0.159 \times 0.31 = \boxed{0.049}$$

$$h(2) = h(4) = hd(2)\,\omega_{Hm}(2) = 0.225 \times 0.77 = \boxed{0.173}$$

$$h(3) = hd(3)\,\omega_{Hm}(3) = 0.25 \times 1 = \boxed{0.25}$$

Step 4: To Find H(z)

$$\boxed{H(z) = \sum_{n=0}^{N-1} h(n)\,z^{-n}}$$

$$= \sum_{n=0}^{6} h(n)\,z^{-n}$$

$$= h(0)z^{0} + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6}$$

$$= 0.006\times 1 + 0.049 z^{-1} + 0.173 z^{-2} + 0.25 z^{-3} + 0.173 z^{-4} + 0.049 z^{-5} + 0.006 z^{-6}$$

$$\boxed{H(z) = 0.006(1+z^{-6}) + 0.049(z^{-1}+z^{-5}) + 0.173(z^{-2}+z^{-4}) + 0.25 z^{-3}}$$

Problem 2:

Design a Ideal Highpass Filter with a frequency response

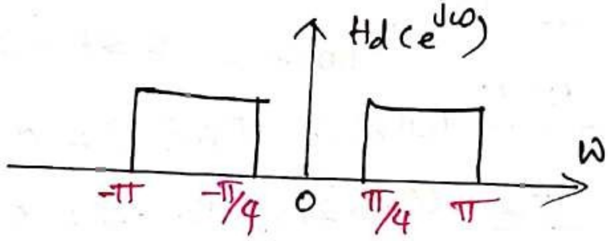
$$H_d(e^{j\omega}) = \begin{cases} 1 & \text{For } \pi/4 \le |\omega| \le \pi \\ 0 & \text{for otherwise} \end{cases}$$

Find the Value of $h(n)$ for $N=11$ using Hanning window Find $H(z)$.

Given: Filter HPF, $N=11$, $\alpha=0$, $\omega_c = \pi/4$

Step 1: To find $H_d(n)$

$h_d(n)$ for Zero phase HPF



$$\boxed{h_d(n) = \frac{\sin \pi n - \sin \omega_c n}{\pi n}}$$

$$h_d(n) = \frac{\sin \pi n - \sin \frac{\pi}{4} n}{\pi n}$$

The desired filter Coefficient for $\boxed{N=11}$

$\boxed{h(n) = h(-n)}$ for zero phase condition

$$h_d(0) = \frac{\sin \pi(0) - \sin \pi/4 \times 0}{\pi \times 0} = \frac{0}{0} \text{ undetermined.}$$

Use L'Hospital Rule $h_d(0) = \frac{\pi - \omega_c}{\pi} = \frac{\pi - \pi/4}{\pi} = \boxed{0.75}$

$$h_d(1) = h_d(-1) = \frac{\sin \pi - \sin \pi/4}{\pi \times 1} = \boxed{-0.225}$$

$$h_d(2) = h_d(-2) = \frac{\sin 2\pi - \sin \pi/2}{\pi \times 2} = \boxed{-0.159}$$

$$hd(3) = hd(-3) = \frac{\sin 3\pi - \sin(3\pi/4)}{3 \times \pi} = \boxed{-0.075}$$

$$hd(4) = hd(-4) = \frac{\sin 4\pi - \sin \pi}{4 \times \pi} = \boxed{0}$$

$$hd(5) = hd(-5) = \frac{\sin 5\pi - \sin 5\pi/4}{5 \times \pi} = \boxed{0.045}$$

Step 2: To find the window sequence

The Non-Causal Hanning window sequence zero phase

$$w_{Hn}(n) = \begin{cases} 0.5 + 0.5\cos\left(\frac{2\pi n}{N-1}\right), & -\left(\frac{N-1}{2}\right) \le n \le \left(\frac{N-1}{2}\right) \\ 0 & \text{otherwise} \end{cases}$$

$$w_{Hn}(n) = \begin{cases} 0.5 + 0.5\cos\left(\frac{2\pi n}{10}\right), & -5 \le n \le 5 \\ 0 & \text{otherwise.} \end{cases}$$

n = 0:

$$w_{Hn}(0) = 0.5 + 0.5\cos\left(\frac{2\pi \times 0}{10}\right) = 0.5 + 0.5 \Rightarrow \boxed{1}$$

n = 1: $w_{Hn}(1) = w_{Hn}(-1) = 0.5 + 0.5\cos(\pi/5) = \boxed{0.904}$

n = 2: $w_{Hn}(2) = w_{Hn}(-2) = 0.5 + 0.5\cos(2\pi/5) = \boxed{0.654}$

n = 3: $w_{Hn}(3) = w_{Hn}(-3) = 0.5 + 0.5\cos(3\pi/5) = \boxed{0.347}$

n = 4: $w_{Hn}(4) = w_{Hn}(-4) = 0.5 + 0.5\cos(4\pi/5) = \boxed{0.094}$

n = 5: $w_{Hn}(5) = w_{Hn}(-5) = 0.5 + 0.5\cos(\pi) = \boxed{0}$

Step 3: To Find h(n)

$$\boxed{h(n) = hd(n)\, w_{Hn}(n)}$$

$$h(0) = hd(0)\, w_{Hn}(0) = 0.75 \times 1 = \boxed{0.75}$$

$$h(1) = h(-1) = hd(1)\, w_{Hn}(1) = 0.225 \times 0.904 = \boxed{-0.203}$$

$$h(2) = h(-2) = hd(2)\, w_{Hn}(2) = -0.159 \times 0.654 = \boxed{-0.104}$$

$$h(3) = h(-3) = hd(3)\, w_{Hn}(3) = -0.075 \times 0.347 = \boxed{-0.026}$$

$$h(4) = h(-4) = hd(4)\, w_{Hn}(4) = 0 \times 0.094 = \boxed{0}$$

$$h(5) = h(-5) = hd(5)\, w_{Hn}(5) = 0.045 \times 0 = \boxed{0}$$

Step 4: To find H(z)

$$H(z) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} h(n) z^{-n}$$

$$= \sum_{n=-5}^{5} h(n) z^{-n}$$

$$= h(-5) z^{5} + h(-4) z^{4} + h(3) z^{3} + h(-2) z^{2} + h(-1) z^{1} + h(0) z^{0}$$
$$+ h(1) z^{-1} + h(2) z^{-2} + h(3) z^{-3} + h(4) z^{-4} + h(5) z^{-5}$$

Let de-arrange and combine

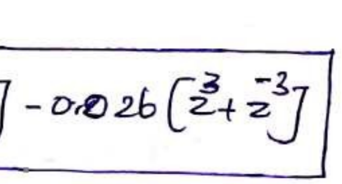
Since $h(n) = h(-n)$

$$= h(0) + h(1)\left[z + z^{-1}\right] + h(2)\left[z^{2} + z^{-2}\right] + h(3)\left[z^{3} + z^{-3}\right] +$$
$$h(4)\left[z^{4} + z^{-4}\right] + h(5)\left[z^{5} + z^{-5}\right]$$

$$\boxed{H(z) = 0.75 - 0.2034\left[z + z^{-1}\right] - 0.1039\left[z^{2} + z^{-2}\right] - 0.026\left[z^{3} + z^{-3}\right]}$$

Problem 3

A BPF of length 7 is required. it is have Lower and upper cut-off frequencies of 3 kHz and 5 kHz respectively. The Sampling frequency is 20 kHz. Determine the filter Co-efficient using Hanning window. Assume the filter to be Causal.



Given: Type of filter : BPF, N = 7

fs (Sampling frequency) = 20 kHz

Cut off frequency $f_{c1} = 3\,kHz$, $f_{c2} = 5\,kHz$

To find ω_{c1}, ω_{c2}

$$W_{c_1} = \frac{2\pi f_{c1}}{f_s} = \frac{2\pi \times 3\times10^3}{20\times10^3} \Rightarrow \boxed{\omega_{c_1} = 0.3\pi \text{ rad/sec}}$$

$$\omega_{c_2} = \frac{2\pi f_{c2}}{f_s} = \frac{2\pi\times5\times10^3}{20\times10^3} \Rightarrow \boxed{\omega_{c_2} = 0.5\pi \text{ rad/sec}}$$

Find the frequency Response:

The desired frequency response of Ideal Band Pass Filter is

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha}, & \omega_{c_1} \le |\omega| \le \omega_{c_2} \\ 0, & \text{otherwise} \end{cases}$$

Where, $\alpha = \frac{N-1}{2} = \frac{7-1}{2} \Rightarrow \boxed{\alpha = 3}$

$$H_d(e^{j\omega}) = \begin{cases} e^{-j3\omega}, & 0.3\pi \le |\omega| \le 0.5\pi \\ 0, & \text{otherwise} \end{cases}$$

Step 1: To Find $h_d(n)$

$$h_d(n) = \frac{\sin\omega_{c_2}(n-\alpha) - \sin\omega_{c_1}(n-\alpha)}{\pi(n-\alpha)}$$

$$\boxed{h_d(n) = \frac{\sin 0.5\pi(n-3) - \sin 0.3\pi(n-3)}{\pi(n-3)}}$$

The desired Filter Co-efficient with N = 7 for Symmetric Condition $\boxed{h(n) = h(N-1-n)}$ is

n = 0: $h(0) = h(6) = \frac{\sin 0.5\pi(0-3) - \sin 0.3\pi(0-3)}{\pi(0-3)}$

$$= \frac{\sin(-1.5\pi) - \sin(-0.9\pi)}{-3\pi} = \frac{1-(-0.3090)}{-3\pi}$$

$$\boxed{h(0) = h(6) = -0.138}$$

n=1: $h(1) = h(5) = \frac{\sin 0.5\pi(1-3) - \sin 0.3\pi(1-3)}{\pi(1-3)}$

$= \frac{\sin(-180) - \sin(-108)}{\pi(-2)} \Rightarrow \frac{0-(-0.9511)}{-2\pi}$

$\boxed{h(1) = h(5) = -0.1514}$

n=2: $h(2) = h(4) = \frac{\sin 0.5\pi(2-3) - \sin 0.3\pi(2-3)}{\pi(2-3)}$

$= \frac{\sin(-90) - \sin(-54)}{-\pi} = \frac{-1+0.8090}{-\pi}$

$\boxed{h(2) = h(4) = 0.0608}$

n=3: $h(3) = \frac{\sin 0.5\pi(3-3) - \sin 0.3\pi(3-3)}{\pi(3-3)} = \frac{0}{0} = \infty$

Use L Hospital Rule

$hd(3) = \frac{\omega c_2 - \omega c_1}{\pi} = \frac{0.5\pi - 0.3\pi}{\pi}$ $\boxed{hd(3) = 0.2}$

Step 2: To find the window co-efficients:

The Hanning window sequence is (Linear phase.

$$\omega Hn(n) = \begin{cases} 0.5 - 0.5\cos\left(\frac{2\pi n}{N-1}\right); & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$\omega Hn(n) = \begin{cases} 0.5 - 0.5\cos\left(\frac{2\pi n}{6}\right) & 0 \le n \le 6 \end{cases}$$

n=0: $\omega Hn(0) = \omega Hn(6) = 0.5 - 0.5\cos\left(\frac{2\pi \times 0}{6}\right) = \boxed{0}$

n=1: $\omega Hn(1) = \omega Hn(5) = 0.5 - 0.5\cos\left(\frac{2\pi \times 1}{6}\right) = \boxed{0.25}$

n=2: $\omega Hn(2) = \omega Hn(4) = 0.5 - 0.5\cos\left(\frac{2\pi \times 2}{6}\right) = \boxed{0.75}$

n=3: $\omega Hn(3) = 0.5 - 0.5\cos\left(\frac{2\pi \times 3}{6}\right) = \boxed{1}$

Step 3: To Find h(n)

$$\boxed{h(n) = h_d(n)\, \omega_{Hn}(n)}$$

$h_d(0) = h_d(6) = h_d(0)\,\omega_{Hn}(0) = -0.138 \times 0 = \boxed{0}$

$h_d(1) = h_d(5) = h_d(1)\,\omega_{Hn}(1) = -0.151 \times 0.25 = \boxed{-0.037}$

$h_d(2) = h_d(4) = h_d(2)\,\omega_{Hn}(2) = 0.060 \times 0.75 = \boxed{0.045}$

$h_d(3) = h_d(3)\,\omega_{Hn}(3) = 0.2 \times 1 = \boxed{0.2}$

Step 4: To Find H(z)

$$\boxed{H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}}$$ [∵ For Linear Phase

$$= \sum_{n=0}^{6} h(n)\, z^{-n}$$

$$= h(0) z^{0} + h(1) z^{-1} + h(2) z^{-2} + h(3) z^{-3} + h(4) z^{-4} + h(5) z^{-5} + h(6) z^{-6}$$

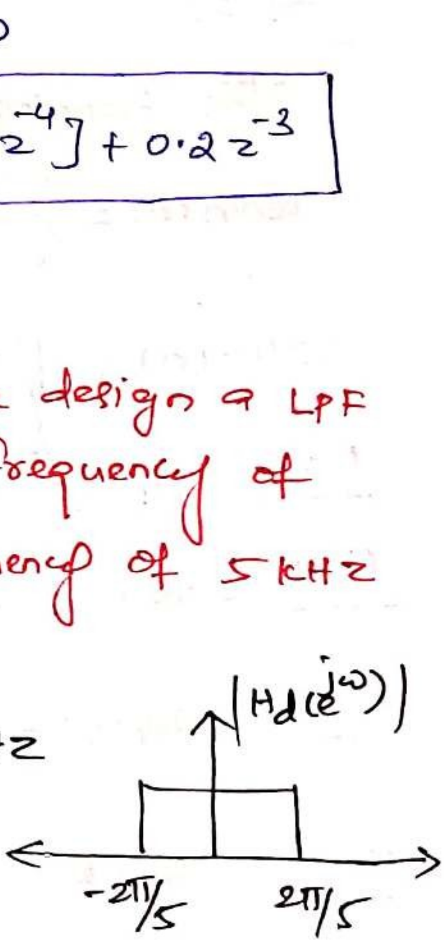
$$= 0 + (-0.037 z^{-1}) + 0.045 z^{-2} + 0.2 z^{-3} + 0.045 z^{-4} + 0.037 z^{-5} + 0$$

$$\boxed{H(z) = -0.037\left[z^{-1} + z^{-5}\right] + 0.045\left[z^{-2} + z^{-4}\right] + 0.2 z^{-3}}$$

Problem 4:

Using Rectangular window technique design a LPF with Passband gain of unity. Cutoff frequency of 1000 Hz and working sampling frequency of 5 KHz the length of impulse is 7.

Given: $N = 7$, Cutoff freq $f_c = 1000\,Hz$
Sampling frequency $f_s = 5\,KHz$
Given filter: LPF



$$\omega_c = \frac{2\pi f_c}{f_s} = \frac{2\pi \times 1000}{5000} \Rightarrow \boxed{\omega_c = \frac{2\pi}{5} \text{ rad/sec}}$$

To Find the desired frequency Response:

The desired frequency response of LPF is

$$H_d(e^{j\omega}) = \begin{cases} 1 & \text{For } -\omega_c \le \omega \le \omega_c \\ 0 & \text{for otherwise} \end{cases}$$

$$H_d(e^{j\omega}) = \begin{cases} 1 & \text{For } -2\pi/5 \le \omega \le 2\pi/5 \\ 0 & \text{For otherwise} \end{cases}$$

Step 1: To Find hd(n)

$$\boxed{h_d(n) = \frac{\sin \omega_c n}{\pi n}}$$ ∵ For zero phase LPF Gain of unity

$$h_d(n) = \frac{\sin\left(\frac{2\pi}{5}\right)n}{\pi n}$$

The desired Filter Co-efficient for symmetric function $\boxed{h(n) = h(-n)}$ for N = 7

n=0:

$$h(0) = \frac{\sin\left(\frac{2\pi}{5}\right)\times 0}{\pi \times 0} = \frac{\sin 0}{0} = \frac{0}{0} = \infty \text{ undetermined}$$

Use 'L' Hospital rule, $h_d(0) = \frac{\omega_c}{\pi} = \frac{2\pi/5}{\pi} = \boxed{0.4}$

n=1:

$$h_d(1) = h_d(-1) = \frac{\sin\left(\frac{2\pi}{5}\right)\times 1}{\pi \times 1} = \boxed{0.302}$$

n=2:

$$h_d(2) = h_d(-2) = \frac{\sin(2\pi/5)\times 2}{\pi \times 2} = \boxed{0.093}$$

n=3:

$$h_d(3) = h_d(-3) = \frac{\sin(2\pi/5)\times 3}{\pi \times 3} = \boxed{-0.062}$$

Step 2: To Find window sequence w(n)

Zerophase, Rectangular window sequence

$$w_R(n) = \begin{cases} 1 & \text{for } -\left(\frac{N-1}{2}\right) \le n \le \left(\frac{N-1}{2}\right) \\ 0 & \text{for otherwise} \end{cases}$$

$$= \begin{cases} 1 & \text{for } -3 \le n \le 3 \\ 0 & \text{for otherwise} \end{cases}$$

$w_R(0) = 1$

$w_R(1) = w_R(-1) = 1$

$w_R(2) = w_R(-2) = 1$

$w_R(3) = w_R(-3) = 1$

Step 3: To find $h(n)$

$$\boxed{h(n) = h_d(n)\, w_R(n)}$$

$h(0) = h_d(0) w_R(0) = 0.4 \times 1 \Rightarrow \boxed{0.4}$

$h(1) = h(-1) = h_d(1) w_R(1) = 0.302 \times 1 = \boxed{0.302}$

$h(2) = h(-2) = h_d(2) w_R(2) = 0.093 \times 1 = \boxed{0.093}$

$h(3) = h(-3) = h_d(3) w_R(3) = -0.062 \times 1 = \boxed{-0.062}$

Step 4: To Find $H(z)$

$$\boxed{H(z) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} h(n) z^{-n}}$$

$$= \sum_{n=-3}^{3} h(n) z^{-n}$$

$$= h(-3) z^{3} + h(-2) z^{2} + h(-1) z^{1} + h(0) z^{0} + h(1) z^{-1} + h(2) z^{-2} + h(3) z^{-3}$$

$$= h(0) + h(1)\left[z^{-1} + z\right] + h(2)\left[z^{-2} + z^{2}\right] + h(3)\left[z^{-3} + z^{3}\right]$$

$$\boxed{H(z) = 0.4 + 0.302(z^{-1} + z) + 0.093(z^{-2} + z^{2}) - 0.062(z^{-3} + z^{3})}$$

Design of Linear-phase FIR Filter:

Frequency Sampling Method:

The desired frequency response of the FIR filter is $|Hd(e^{j\omega})|$.

Step 1: The discrete Fourier Transform $H(k)$ can be obtained by sampling frequency response $Hd(e^{j\omega})$ at N-points $\boxed{H(k) = Hd(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}}}$

Step 2: The filter Co-efficients can be obtained by using IDFT $h(n) = \text{IDFT}\{H(k)\}$

For Linear phase FIR filter, the $h(n)$ must satisfy the symmetry condition $\boxed{h(n) = h(N-1-n)}$; $n = 0, 1, 2 \cdots N-1$

For $N \rightarrow$ odd

$$\boxed{h(n) = \frac{1}{N}\left\{H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} \mathrm{Re}\left[H(k)e^{\frac{j2\pi kn}{N}}\right]\right\}}$$

For $N \rightarrow$ Even:

$$\boxed{h(n) = \frac{1}{N}\left\{H(0) + 2\sum_{k=1}^{N/2-1} \mathrm{Re}\left[H(k)e^{\frac{j2\pi kn}{N}}\right]\right\}}$$

Problem:

Determine the Filter Co-efficient $h(n)$ obtained by Sampling the given frequency response for $N = 7$

$$Hd(e^{j\omega}) = \begin{cases} e^{-j\frac{(N-1)\omega}{2}}, & 0 \le |\omega| \le \pi/2 \\ 0, & \pi/2 \le |\omega| \le \pi \end{cases}$$

Soln: Given data: $N=7$

$$H_d(e^{j\omega}) = \begin{cases} e^{\frac{-j(7-1)\omega}{2}}, & 0 \le |\omega| \le \pi/2 \\ 0, & \pi/2 \le |\omega| \le \pi \end{cases}$$

$$H_d(e^{j\omega}) = \begin{cases} e^{-j3\omega}, & 0 \le |\omega| \le \pi/2 \\ 0, & \text{otherwise} \end{cases}$$

The given filter is a symmetric filter at $\frac{N-1}{2}$

$$\frac{7-1}{2} = 3$$

Step 1: Discrete Fourier Transform $H(k)$

$$H(k) = H_d(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}} = \frac{2\pi k}{7}$$

$$H(k) = \begin{cases} e^{-j3\left(\frac{2\pi k}{7}\right)} ; & 0 \le \left|\frac{2\pi k}{7}\right| \le \pi/2 \\ 0 & \text{otherwise} \end{cases}$$

$$H(k) = \begin{cases} e^{-j\left(\frac{6\pi k}{7}\right)} & ; 0 \le k \le 7/4 \\ 0 & ; \text{otherwise} \end{cases}$$

$$\because \frac{2\pi k}{7} = \frac{\pi}{2}$$

$$k = \pi/2 \times \frac{7}{2\pi}$$

$$\boxed{k = 7/4}$$

$$k = 1.75$$

$$k = 0, 1$$

k = 0, 1 :

$$H(0) = e^{-j\left(\frac{6\pi \times 0}{7}\right)} = e^{0} = \boxed{1 = H(0)}$$

$$H(1) = e^{-j\left(\frac{6\pi \times 1}{7}\right)} = \boxed{e^{-j6\pi/7} = H(1)}$$

$$\because H(2) = H(3) = H(4) = H(5) = H(6) = 0$$

Filter coefficients of the Linear phase FIR filter $h(n)$:

$$h(n) = h(N-1-n)$$

$h(0) = h(7-1-0) = h(6)$

$h(1) = h(7-1-1) = h(5)$

$h(2) = h(7-1-2) = h(4)$

$h(3) = h(7-1-3) = h(3)$

For N = 7 (odd)

$$h(n) = \frac{1}{N}\left\{ H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} \mathrm{Re}\left[H(k)\, e^{\frac{j2\pi kn}{N}}\right]\right.$$

$n = 0, 1, 2 \cdots N-1$

$$h(n) = \frac{1}{7}\left\{ H(0) + 2\sum_{k=1}^{\frac{7-1}{2}} \mathrm{Re}\left[H(k)\, e^{\frac{j2\pi kn}{7}}\right]\right\}$$

$n = 0, 1, 2, 3, 4, 5, 6$

$$= \frac{1}{7}\left\{ H(0) + 2\left[\sum_{k=1}^{3} \mathrm{Re}\left[H(k)\, e^{\frac{j2\pi kn}{7}}\right]\right]\right\}$$

$$= \frac{1}{7}\left\{ H(0) + 2\left[\mathrm{Re}\left(H(1)\, e^{\frac{j2\pi 1 n}{7}}\right) + \mathrm{Re}\left(H(2)\, e^{\frac{j2\pi\times 2\times n}{7}}\right) + \mathrm{Re}\left(H(3)\, e^{\frac{j2\pi\times 3\times n}{7}}\right)\right]\right.$$

$$= \frac{1}{7}\left\{ 1 + 2\left[\mathrm{Re}\left(e^{-j6\pi/7} \cdot e^{j2\pi n/7}\right) + 0 + 0\right]\right\}$$

$$= \frac{1}{7}\left\{ 1 + 2\,\mathrm{Re}\left[e^{+j\frac{2\pi(n-3)}{7}}\right]\right\}$$

$$e^{j\theta} = \underbrace{\cos\theta}_{\text{Real}} + j\underbrace{\sin\theta}_{\text{Imaginary}}$$

$$\boxed{h(n) = \frac{1}{7}\left\{1 + 2\cos\left(\frac{2\pi(n-3)}{7}\right)\right\} \quad n = 0, 1, 2, 3, 4, 5, 6}$$

n = 0; $h(0) = h(6) = \frac{1}{7}\left\{1 + 2\cos\left(\frac{2\pi(0-3)}{7}\right)\right\}$

$$= \frac{1}{7}\left\{1 + 2\cos\left(-\frac{6\pi}{7}\right)\right\} \Rightarrow \frac{1}{7}\left[-0.8019\right]$$

$$\boxed{h(0) = h(6) = -0.1146}$$

n = 1: $h(1) = h(5) = \frac{1}{7}\left\{1 + 2\cos\left(\frac{2\pi(1-3)}{7}\right)\right\}$

$$= \frac{1}{7}\left\{1 + 2\cos\left(-\frac{4\pi}{7}\right)\right\} = \frac{1}{7}\left\{1 + 2(-0.2225)\right\}$$

$$\boxed{h(1) = h(5) = 0.0793}$$

$n=2,\quad h(2)=h(4)=\frac{1}{7}\left\{1+2\cos\left(\frac{2\pi(2-3)}{7}\right)\right\}$

$$=\frac{1}{7}\left\{1+2\cos\left(-\frac{2\pi}{7}\right)\right\}=\frac{1}{7}\left\{1+2(0.6235)\right\}$$

$$\boxed{h(2)=h(4)=0.321}$$

$n=3:\quad h(3)=\frac{1}{7}\left\{1+2\cos\left(\frac{2\pi(2-3)}{7}\right)\right\}$

$$=\frac{1}{7}\left\{1+2\cos 0\right\}\Rightarrow\boxed{h(3)=0.4286}$$

Filter Coefficients:

$$h(n)=\{\underset{\uparrow}{-0.1146},\ 0.0793,\ 0.321,\ 0.428,\ 0.321,\ 0.0793,\ -0.1146\}$$

Problem:

Determine the Coefficients of a Linear phase FIR Filter of length $n=15$, which has a symmetrical unit sample response that satisfies the conditions.

$$H\left(\frac{2\pi k}{15}\right)=1,\ \text{for } k=0,1,2,3$$

$$=0.4;\ \text{for } k=4$$

$$=0,\ \text{for } k=5,6,7$$

Soln:

Given Data: $n=15$ $\qquad W=\omega_k=\frac{2\pi k}{N}=\frac{2\pi k}{15}$

$$\alpha=\frac{N-1}{2}=\frac{15-1}{2}\Rightarrow\boxed{\alpha=7}$$

Step 1: The discrete Fourier Transform H(k)

$$H(k)=H_d(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{N}=\frac{2\pi k}{15}}$$

The desired frequency response

$$H(K) = \begin{cases} 1 * e^{-j\alpha\omega_K}, & K=0,1,2,3 \\ 0.4 * e^{-j\alpha\omega_k} & \text{for } k=4 \\ 0 & \text{for } k=5,6,7 \end{cases} \Rightarrow \begin{cases} e^{-j7\left(\frac{2\pi k}{15}\right)}, & k=0,1,2,3 \\ 0.4\, e^{-j7\left(\frac{2\pi k}{15}\right)}, & k=4 \\ 0 , & k=5,6,7 \end{cases}$$

Step 2: for $N=7$ (odd)

$$h(n) = \frac{1}{N}\left\{ H(0) + 2\sum_{k=1}^{\left(\frac{N-1}{2}\right)} \operatorname{Re}\left[H(k)\, e^{\frac{j2\pi kn}{N}} \right] \right\} \quad n=0,1\ldots N-1$$

$$= \frac{1}{15}\left\{ H(0) + 2\sum_{K=1}^{7} \operatorname{Re}\left[e^{-j\left(\frac{14\pi k}{15}\right)} e^{j\left(\frac{2\pi kn}{N}\right)} \right] \right\}$$

$$= \frac{1}{15}\left\{ 1 + 2\sum_{k=1}^{7} \operatorname{Re}\left[e^{\frac{j2\pi k(n-7)}{15}} \right] \right\}$$

$$= \frac{1}{15}\left\{ 1 + 2\sum_{k=1}^{7} \cos \frac{2\pi k}{15}(n-7) \right\}$$

$$= \frac{1}{15}\left\{ 1 + 2\left[\cos\frac{2\pi}{15}(n-7) + \cos\frac{4\pi}{15}(n-7) + \cos\frac{6\pi}{15}(n-7) + 0.4\cos\frac{8\pi}{15}(n-7) + 0 + 0 + 0 \right] \right\}$$

$$\boxed{h(n) = \frac{1}{15}\left\{ 1 + 2\left[\cos\frac{2\pi}{15}(n-7) + \cos\frac{4\pi}{15}(n-7) + \cos\frac{6\pi}{15}(n-7) + 0.4\cos\frac{8\pi}{15}(n-7) \right] \right\}}$$

when $\underline{n=0}$, $$h(0) = \frac{1}{15}\left[1 + 2\left(\cos\frac{2\pi}{15}(0-7) + \cos\frac{4\pi}{15}(0-7) + \cos\frac{6\pi}{15}(0-7) + 0.4\cos\frac{8\pi}{15}(0-7) \right) \right]$$

$$= \frac{1}{15}\left[1 + 2\left[-0.978 + 0.9135 - 0.8090 + 0.2677 \right] \right]$$

$$= \frac{1}{15}\left[1 + 2(-0.6058) \right] \Rightarrow \boxed{h(0) = -0.0141}$$

when $\underline{n=1}$: $h(1) = \frac{1}{15}\left[1+2\left(\cos\frac{2\pi}{15}(1-7)+\cos\frac{4\pi}{15}(1-\right.\right.$

$$+\cos\frac{6\pi}{15}(1-7)+0.4\cos\frac{8\pi}{15}(\text{[illegible]}-7)\Big)\Big]$$

$$= \frac{1}{15}\left[1+2\left\{-0.8090+0.3090+0.3090-0.3236\right\}\right]$$

$$= \frac{1}{15}\left[1+2\times-0.5146\right] \Rightarrow \boxed{h(1)=-0.0019}$$

$\underline{\text{when, } n=2:}$

$$h(2)=\frac{1}{15}\Big\{1+2\Big[\cos\frac{2\pi}{15}(2-7)+\cos\frac{4\pi}{15}(2-7)+\cos\frac{6\pi}{15}(2-7)$$
$$+0.4\cos\frac{8\pi}{15}(2-7)\Big]\Big\}$$

$$=\frac{1}{15}\left\{1+2\left[-0.5-0.5+1-0.2\ldots\right]\right\}$$

$$=\frac{1}{15}\left[1+2\times0.2\ldots\right] = \boxed{h(2)=0.04}$$

$\underline{\text{when } n=3,}$

$$h(3)=\frac{1}{15}\Big\{1+2\Big[\cos\frac{2\pi}{15}(3-7)+\cos\frac{4\pi}{15}(3-7)+\cos\frac{6\pi}{15}(3-7)$$
$$+0.4\cos\frac{8\pi}{15}(3-7)\Big]\Big\}$$

$$=\frac{1}{15}\left\{1+2\left[-0.1045-0.9781+0.3090+0.3654\right]\right\}$$

$$=\frac{1}{15}\left\{1+2\times-0.4082\right\} \Rightarrow \boxed{h(3)=0.0122}$$

$\underline{\text{when } n=4:}$

$$h(4)=\frac{1}{15}\Big\{1+2\Big[\cos\frac{2\pi}{15}(4-7)+\cos\frac{4\pi}{15}(4-7)+\cos\frac{6\pi}{15}(4-7)$$
$$+0.4\cos\frac{8\pi}{15}(4-7)\Big]\Big\}$$

$$=\frac{1}{15}\left\{1+2\left[0.3090-0.8090-0.8090+0.1236\right]\right\}$$

$$=\frac{1}{15}\left\{1+2\times-1.1854\right\} \Rightarrow \boxed{h(4)=-0.0914}$$

$\underline{\text{when } n=5}$

$$h(5)=\frac{1}{15}\Big\{1+2\Big[\cos\frac{2\pi}{15}(5-7)+\cos\frac{4\pi}{15}(5-7)+\cos\frac{6\pi}{15}(5-7)$$
$$+0.4\cos\frac{8\pi}{15}(5-7)\Big]\Big\}$$

$$= \frac{1}{15}\left\{1+2\left[0.6691-0.1045-0.8090-0.3913\right]\right\}$$

$$= \frac{1}{15}\left[1+2\times -0.6357\right] \Rightarrow \boxed{h(5) = -0.0181}$$

when $n=6$;

$$h(6) = \frac{1}{15}\left\{1+2\left[\cos\frac{2\pi}{15}(6-7)+\cos\frac{4\pi}{15}(6-7)+\cos\frac{6\pi}{15}(6-7) + 0.4\cos\frac{8\pi}{15}(6-7)\right]\right\}$$

$$= \frac{1}{15}\left[1+2\left[0.9135+0.6691+0.3090-0.0418\right]\right\}$$

$$= \frac{1}{15}\left[1+2\times 1.8498\right] \Rightarrow \boxed{h(6) = 0.3133}$$

when $n=7$,

$$h(7) = \frac{1}{15}\left[1+2\left(\cos\frac{2\pi}{15}(7-7)+\cos\frac{4\pi}{15}(7-7)+\cos\frac{6\pi}{15}(7-7) + 0.4\cos\frac{8\pi}{15}(7-7)\right]\right.$$

$$= \frac{1}{15}\left[1+2\times(1+1+1+0.4)\right]$$

$$\Rightarrow \frac{1}{15}\left[1+2\times 3.4\right] \qquad \boxed{h(7) = 0.52}$$

Using symmetry condition $h(n) = h(N-1-n)$

when $n=8$, $h(8) = h(15-1-8) = h(6) = 0.3133$

$n=9$, $h(9) = h(15-1-9) = h(5) = -0.0187$

$n=10$, $h(10) = h(15-1-10) = h(4) = -0.0914$

$n=11$, $h(11) = h(15-1-11) = h(3) = 0.0122$

$n=12$, $h(12) = h(15-1-12) = h(2) = 0.04$

$n=13$, $h(13) = h(15-1-13) = h(1) = -0.0019$

$n=14$, $h(14) = h(15-1-14) = h(0) = -0.0141$

Step 3: the Transfer Function $H(z)$

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}$$

$$= \sum_{n=0}^{14} h(n) z^{-n}$$

$$= h(0)z^{-0} + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6} + h(7)z^{-7} + h(8)z^{-8} + h(9)z^{-9} + h(10)z^{-10} + h(11)z^{-11} + h(12)z^{-12} + h(13)z^{-13} + h(14)z^{-14}$$

Using symmetry condition,

$$\boxed{\begin{aligned} H(z) = & -0.0141\left[z^{0}+z^{-14}\right] - 0.0019\left[z^{-1}+z^{-13}\right] + 0.04\left[z^{-2}+z^{-12}\right] \\ & + 0.0122\left[z^{-3}+z^{-11}\right] - 0.0914\left[z^{-4}+z^{-10}\right] - 0.0181\left[z^{-5}+z^{-9}\right] \\ & + 0.3130\left[z^{-2}+z^{-8}\right] + 0.52\left[z^{-7}\right] \end{aligned}}$$

Problem:

Using frequency sampling method, design a bandpass filter with the following specification sampling frequency = 8000 Hz, cutoff frequencies $F_{c_1} = 1000$ Hz, $f_{c_2} = 3000$ Hz, Determine the Filter coefficients for N=7.

Soln: Given: $N=7$, $f_s = 8000$ Hz, $f_{c_1} = 1000$ Hz, $f_{c_2} = 3000$ Hz.

$$\omega_{c_1} = \frac{2\pi f_{c_1}}{f_s} = \frac{2\pi \times 1000}{8000} \Rightarrow \boxed{\omega_{c_1} = 0.2\pi}$$

$$\omega_{c_2} = \frac{2\pi f_{c_2}}{f_s} = \frac{2\pi \times 3000}{8000} \Rightarrow \boxed{\omega_{c_2} = 0.75\pi}$$

Step 1: The desired frequency response of BPF

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha}, & \omega_{c_1} \le \omega \le \omega_{c_2} \\ 0, & \text{otherwise} \end{cases}$$

$$\alpha = \frac{N-1}{2} \Rightarrow \frac{7-1}{2} \quad \boxed{\alpha = 3}$$

where,

$$H_d(e^{j\omega}) = \begin{cases} e^{-j3\omega}; & 0.25\pi \le \omega \le 0.75\pi \\ 0, & \text{otherwise.} \end{cases}$$

Step 2: The discrete Fourier Transform H(k)

$$H(k) = H_d(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}}$$

$$= \begin{cases} e^{-j3\frac{2\pi k}{7}}, & k=1,2 \\ 0 & \text{otherwise} \end{cases} \Rightarrow \begin{cases} e^{-\frac{6\pi k}{7}} & : k=1,2 \\ 0 & \text{o.w} \end{cases}$$

Step 3: To find h(n) for N odd.

$$h(n) = \frac{1}{N}\left\{ H(0) + 2\sum_{k=1}^{\left(\frac{N-1}{2}\right)} \mathrm{Re}\left[H(k)\, e^{\frac{j2\pi kn}{N}}\right]\right\}$$

$$h(n) = \frac{1}{7}\left\{0 + 2\sum_{k=1}^{3} \mathrm{Re}\left[e^{-j\frac{6\pi k}{7}}\, e^{\frac{j2\pi kn}{7}}\right]\right\}$$

$$= \frac{1}{7}\left\{0 + 2\sum_{k=1}^{2} \mathrm{Re}\left[e^{j\frac{2\pi k}{7}(n-3)}\right]\right\}$$

$$h(n) = \frac{1}{7}\left\{2\sum_{k=1}^{2} \cos\frac{2\pi k}{7}(n-3)\right\}$$

$$\boxed{h(n) = \frac{1}{7}\left[2\cos\frac{2\pi}{7}(n-3) + 2\cos\frac{4\pi}{7}(n-3)\right]}$$

Use symmetry condition, $h(n) = h(N-1-n)$

When, n=0, $h(0) = h(6) = \frac{1}{7}\left[2\cos\left[\frac{2\pi}{7}(0-3)\right] + 2\cos\frac{4\pi}{7}(0-3)\right]$

$$= \frac{1}{7}\left[-1.8019 + 1.247\right] \Rightarrow \boxed{h(0) = h(6) = -0.079}$$

When n=1, $h(1) = h(5) = \frac{1}{7}\left[2\cos\frac{2\pi}{7}(1-3) + 2\cos\frac{4\pi}{7}(1-3)\right]$

$$= \frac{1}{7}\left[-0.445 - 1.8019\right] \Rightarrow \boxed{h(1) = h(5) = -0.320}$$

When n=2, $h(2) = h(4) = \frac{1}{7}\left[2\cos\frac{2\pi}{7}(2-3) + 2\cos\frac{4\pi}{7}(2-3)\right]$

$$= \frac{1}{7}\left[1.2469 - 0.445\right] \Rightarrow \boxed{h(2) = h(4) = 0.114}$$

When n=3, $h(3) = \frac{1}{7}\left[2\cos\frac{2\pi}{7}(3-3) + 2\cos\frac{4\pi}{7}(3-3)\right]$

$$= \frac{1}{7}\left[2+2\right] \Rightarrow \boxed{h(3) = 0.571}$$

Step 4: The Transfer function H(z)

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

$$= \sum_{n=0}^{6} h(n) z^{-n}$$

$$= h(0) z^{0} + h(1) z^{-1} + h(2) z^{-2} + h(3) z^{-3} + h(4) z^{-4} + h(5) z^{-5} + h(6) z^{-6}$$

$$\boxed{H(z) = -0.079\left[1+z^{-6}\right] - 0.32\left[z^{-1}+z^{-5}\right] + 0.164\left[z^{-2}+z^{-4}\right] + 0.571 z^{-3}}$$

DESIGN OF FIR FILTER USING FOURIER SERIES METHOD:

The desired frequency response of FIR filter can be represented by the Fourier Series.

$$\boxed{H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d(n) e^{-j\omega n}}$$

Design steps:

Step 1: For the desired frequency response $H_d(e^{j\omega})$, find the desired impulse sequence of the filter

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) \cdot e^{j\omega n} \cdot d\omega \,, \quad -\omega \leq n \leq \omega$$

Step 2: Truncate $h_d(n)$ at $n = \pm\left(\frac{N-1}{2}\right)$ to get the finite duration sequence h(n)

$$h(n) = \begin{cases} h_d(n) \;; & |n| \leq \frac{N-1}{2} \\ 0 \;, & \text{otherwise} \end{cases}$$

For a symmetrical Impulse response, symmetric at n=0 $\quad h(-n) = h(n)$

Step 3: To find the transfer function H(z) of the realizable filter

$$H(z) = z^{-\frac{(N-1)}{2}} \cdot \left[h(0) + \sum_{n=1}^{(N-1)/2} h(n) \left[z^{n} + z^{-n} \right] \right]$$

Disadvantage of Fourier Series Method:

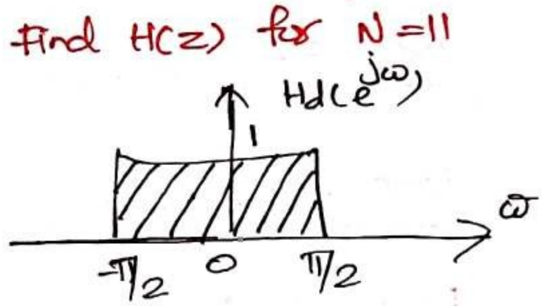
(i) The direct truncation of the series leads to oscillation both in passband and stopband. this is known as Gibb's phenomenon.

(ii) Reduce the oscillations, the Filter coefficients are multiplied by a window frequency in Finite duration.

Problem:

Design an Ideal Lowpass filter with a frequency response

$$H_d(e^{j\omega}) = \begin{cases} 1 & \text{for } -\pi/2 \le \omega \le \pi/2 \\ 0 & \text{for otherwise} \end{cases}$$ Find $H(z)$ for $N=11$



Soln: Given $N=11$, $\omega_c = \pi/2$

The given Filter is symmetric about $\alpha=0$

Step 1: Desired Impulse sequence of the filter

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega.$$

$$= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} 1 \cdot e^{j\omega n} d\omega \Rightarrow \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi/2}^{\pi/2}$$

$$\Rightarrow \frac{1}{2\pi j n}\left[e^{j\pi/2 n} - e^{-j\pi/2 n}\right] \Rightarrow \frac{1}{\pi n}\left[\frac{e^{-j\pi/2 n} - e^{-j\pi/2 n}}{2j}\right]$$

$$\boxed{h_d(n) = \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right)}$$

(or)

$h_d(n)$ for zero phase LPF

$$h_d(n) = \frac{\sin \omega_c n}{n\pi} \qquad \because \omega_c = \pi/2$$

$$\boxed{h_d(n) = \frac{\sin \pi/2\, n}{n\pi}}$$

Step 2: Truncate the $h_d(n)$ to N=11 Samples.

$$h(n) = \begin{cases} \frac{1}{n\pi} \sin(n\pi/2) & ; \; |n| \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

For n=0, h(n) becomes Intermediate

$$h(0) = \frac{1}{0} \sin 0 = \frac{0}{0} \Rightarrow \text{Intermediate}$$

$$h(0) = \frac{\omega_c}{\pi} \Rightarrow \frac{\pi/2}{\pi} = \frac{1}{2} \Rightarrow \boxed{h(0) = 0.5}$$

For n=1, $h(1) = h(-1) = \frac{\sin \pi/2}{\pi} = \frac{1}{\pi} \Rightarrow \boxed{h(1) = h(-1) = 0.3183}$

For n=2, $h(2) = h(-2) = \frac{\sin 2\pi/2}{2\pi} = \frac{\sin \pi}{\pi} \Rightarrow \boxed{h(2) = h(-2) = 0}$

For n=3, $h(3) = h(-3) = \frac{\sin 3\pi/2}{3\pi} = -\frac{1}{3\pi} \Rightarrow \boxed{h(3) = h(-3) = -0.106}$

For n=4, $h(4) = h(-4) = \frac{\sin 4\pi/2}{4\pi} = \frac{\sin 2\pi}{4\pi} \Rightarrow \boxed{h(4) = h(-4) = 0}$

For n=5 $h(5) = h(-5) = \frac{\sin 5\pi/2}{5\pi} = 1/5\pi \Rightarrow \boxed{h(5) = h(-5) = 0.06366}$

Step 3: Find the transfer function H(z)

$$H(z) = z^{-\left(\frac{N-1}{2}\right)} \left[h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n) \left[z^{n} + z^{-n} \right] \right]$$

$$= z^{-\left(\frac{11-1}{2}\right)} \left[0.5 + \sum_{n=1}^{5} h(n) \left[z^{n} + z^{-n} \right] \right]$$

$$= z^{-5} \Big[0.5 + h(1) \left[z^{1} + z^{-1} \right] + h(2) \left[z^{2} + z^{-2} \right] + h(3) \left[z^{3} + z^{-3} \right] + h(4) \left[z^{4} + z^{-4} \right] + h(5) \left[z^{5} + z^{-5} \right] \Big]$$

$$= z^{-5} \Big[0.5 + 0.3183 (z^{1} + z^{-1}) + 0 + (-0.106) (z^{3} + z^{-3}) + 0 + 0.06366 (z^{5} + z^{-5}) \Big]$$

$$= z^{-5} \Big[0.5 + 0.3183 z^{-1} + 0.3183 z^{-1} - 0.106 z^{3} - 0.106 z^{-3} + 0.06366 z^{5} + 0.06366 z^{-5} \Big]$$

$$= 0.5z^{-5}+0.3183z^{-4}+0.3183z^{-6}-0.106z^{-2}-0.106z^{-8}+0.06366z^{0}+0.06366z^{-10}$$

Transfer function of FIR filter

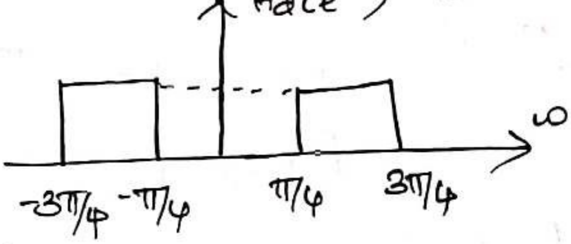
$$\boxed{H(z)=\left\{0.06366-0.106z^{-2}+0.3183z^{-4}+0.5z^{-5}+0.3183z^{-6}-0.106z^{-8}+0.06366z^{-10}\right\}}$$

Filter coefficients:

$$h(n)=\left\{0.06366, 0, -0.106, 0, 0.3183, 0.5, 0.3183, 0, -0.106, 0, 0.06366\right\}$$

Problem 2:

Design an Ideal bandpass filter with a frequency response $H_d(e^{j\omega}) = \begin{cases} 1, & \pi/4 \le |\omega| \le 3\pi/4 \\ 0 & \text{otherwise} \end{cases}$ Find $H(z)$ for $N=11$.



Soln: Given: $N=11$

Step 1: Desired Impulse sequence $h_d(n)$:

Given filter BPF, zero phase $\alpha=0$,

$$h_d(n)=\frac{\sin\omega_{c_2}n-\sin\omega_{c_1}n}{\pi n}$$

$$\boxed{h_d(n)=\frac{\sin\left(\frac{3\pi}{4}n\right)-\sin\left(\frac{\pi}{4}n\right)}{\pi n}} \quad -5\le n\le 5$$

Step 2: Truncate $h_d(n)$ to $N=11$ samples:

For symmetric filter with odd N: $h(n)=h(-n)$

$$h(n)=\begin{cases} h_d(n) & ; |n|\le \frac{N-1}{2} \\ 0 & , \text{otherwise}\end{cases}$$

$$h(n)=\begin{cases} \frac{1}{n\pi}\left(\sin\frac{3\pi}{4}n-\sin\frac{\pi}{4}n\right. & ; |n|\le 5 \\ 0 & ; \text{otherwise}\end{cases}$$

for $n=0$, $h(0) = \frac{1}{0}(\sin 0 - \sin 0) = \frac{0}{0}$ = Intermediate.

use 'L' Hospital Rule,

$$h(0) = \frac{\omega c_2 - \omega c_1}{\pi} = \frac{3\pi/4 - \pi/4}{\pi} \Rightarrow \frac{\frac{2\cancel{\pi}}{4}}{\cancel{\pi}} \Rightarrow \frac{1}{2} \Rightarrow \boxed{h(0) = 0.5}$$

n=1: $$h(1) = h(-1) = \frac{\sin\frac{3\pi}{4} - \sin\pi/4}{\pi} = \frac{0.707 - 0.707}{\pi}$$

$$\boxed{h(1) = h(-1) = 0}$$

n=2: $$h(2) = h(-2) = \frac{\sin\frac{6\pi}{4} - \sin\frac{2\pi}{4}}{2\pi} = \frac{-1-1}{2\pi} \Rightarrow \frac{-\cancel{2}}{\cancel{2}\pi}$$

$$\boxed{h(2) = h(-2) = -0.3183}$$

n=3: $$h(3) = h(-3) = \frac{\sin\frac{9\pi}{4} - \sin\frac{3\pi}{4}}{3\pi} \Rightarrow \frac{0.707 - 0.707}{3\pi} = 0$$

$$\boxed{h(3) = h(-3) = 0}$$

n=4: $$h(4) = h(-4) = \frac{\sin\frac{12\pi}{4} - \sin\frac{4\pi}{4}}{4\pi} = \frac{\sin 3\pi - \sin\pi}{4\pi}$$

$$\boxed{h(4) = h(-4) = 0}$$

n=5: $$h(5) = h(-5) = \frac{\sin\frac{15\pi}{4} - \sin\frac{5\pi}{4}}{5\pi} = \frac{-0.707 + 0.707}{5\pi}$$

$$\boxed{h(5) = h(-5) = 0}$$

Step 8: Transfer Function of FIR Filter H(z)

$$H(z) = z^{-\left(\frac{N-1}{2}\right)}\left[h(0) + \sum_{n=1}^{\left(\frac{N-1}{2}\right)} h(n)\left[z^{n} + z^{-n}\right]\right]$$

$$= z^{-\left(\frac{11-1}{2}\right)}\left[h(0) + \sum_{n=1}^{5} h(n)\left[z^{n} + z^{-n}\right]\right]$$

$$= z^{-5}\Big[h(0) + h(1)\left[z^{1} + z^{-1}\right] + h(2)\left[z^{2} + z^{-2}\right] + h(3)\left[z^{3} + z^{-3}\right]$$
$$+ h(4)\left[z^{4} + z^{-4}\right] + h(5)\left[z^{5} + z^{-5}\right]\Big].$$

$$= z^{-5}\left[0.5+0-0.3183(z^{2}+z^{-2})+0+0+0\right]$$

$$= z^{-5}\left[0.5-0.3183\,z^{2}-0.3183\,z^{-2}\right]$$

$$= 0.5z^{-5}-0.3183z^{-3}-0.3183\,z^{-7}$$

Transfer function of the realizable Bandpass FIR filter

$$\boxed{H(z) = -0.8183\,z^{-3}+0.5z^{-5}-0.3183\,z^{-7}}$$

Filter coefficient:

$$h(n) = \{\underset{\uparrow}{0}, 0, 0, -0.3183, 0, 0.5, 0, 0.8183, 0\}$$

Fixed point and floating point number representation ADC - Quantization - Truncation and rounding - Quantization noise - Input/output Quantization - Coefficient Quantization Error - product quantization Error - overflow Error - Limit cycle osuillations due to product quantization and Summation - Scaling to prevent overflow.

Number Representation:

In DSP, a number 'N' can be reprecented to any decired format Using number System. To represent the numbers in any digital Hardware.

Types of Number Representation:

(I) Fixed point Representation
- Sign Magnitude Form
- 1's Complement Form
- 2's Complement Form

(ii) Floating point Representation

(iii) Block Floating point Representation.

(I) Fixed point Representation:

The position of the Binary point is fixed.

Example: In Binary

$$\underbrace{11}_{\text{Integer part}}\ \underset{\uparrow\ \text{Point}}{.}\ \underbrace{01011}_{\text{Fractional part}}$$

In Decimal,

$$\underset{\text{Integer part}}{3}\ .\ \underbrace{34375}_{\text{Fractional part}}$$

Representation of Negative numbers in Fixed point

Algorithm: (i) sign-magnitude form
(ii) One's Complement form
(iii) Two's Complement form.

(i) Sign-Magnitude Form:

→ The Most significant bit (MSB) is set to
1 → to represent the Negative sign
0 → to represent the positive sign

Example① $(-1.25)_{10} \Rightarrow (\underset{\overset{\uparrow}{\text{-ve sign}}}{1}1.01)_2$

$(+1.25)_{10} \Rightarrow (\underset{\overset{\uparrow}{\text{+ve sign}}}{0}1.01)_2$

$$\begin{array}{r} .25 \\ 2 \\ \hline 0.50 \\ 2 \\ \hline 1.00 \end{array} \downarrow \qquad 01 \leftarrow .25$$

Example② $(0.125)_{10}$

$$\begin{array}{r} 0.125 \\ 2 \\ \hline 0.250 \\ 2 \\ \hline 0.500 \\ 2 \\ \hline 1.000 \\ 2 \\ \hline 0.000 \end{array} \downarrow$$

$(0.125)_{10} = (0.0010)_2$

For $+(0.125)_{10} = (0.0010)_2$

For $-(0.125)_{10} = (1.0010)_2$

(ii) One's Complement Form:

In this method, the negative number is obtained by complementing all the bits.

Example: $(0.875)_{10}$

$$\begin{array}{r} .875 \\ 2 \\ \hline 1.750 \\ 2 \\ \hline 1.500 \\ 2 \\ \hline 1.000 \\ 2 \\ \hline 0.000 \end{array} \downarrow$$

$(0.875)_{10} \Rightarrow (\underset{\text{+ve}}{0}.\underset{\downarrow\downarrow\downarrow\downarrow}{1110})_2$ ↓ 1s complement

$-(0.875)_{10} \Rightarrow (\underset{\text{-ve sign}}{1}.0001)_2$

Example ② $(0.125)_{10}$

$$\begin{array}{r} .125 \\ \times 2 \\ \hline 0.250 \\ \times 2 \\ \hline 0.500 \\ \times 2 \\ \hline 1.000 \\ \times 2 \\ \hline 0.000 \end{array}$$

$(0.125)_{10} = (0.0010)_2$

$-(0.125)_{10}$ = Complement of $(0.0010)_2$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \downarrow\downarrow\downarrow\downarrow$ 1s complement

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad 1101$

$(-0.125)_{10} = (1.1101)_2$

(iii) 2's Complement Form :-

→ The negative number can be obtained by complementing all the bits and adding one to the least significant bit.

Example ① $(0.875)_{10}$

$$\begin{array}{r} .875 \\ \times 2 \\ \hline 1.750 \\ \times 2 \\ \hline 1.500 \\ \times 2 \\ \hline 1.000 \\ \times 1 \\ \hline 0.000 \end{array}$$

$(0.875)_{10} = (0.1110)_2$ (1's)

$\downarrow\ \downarrow\downarrow\downarrow\downarrow$ ← Complementing each bit

1.0001

$\underline{\quad\quad\quad 1}$ (+) Adding 1 bit (2's)

$(-0.875)_{10}$ $\quad 1.0010$

Example ② $(0.125)_{10}$

$(0.125)_{10} = (0.0010)_2$ (1's)

$\downarrow\ \downarrow\downarrow\downarrow\downarrow$ ← complementing each bit

1.1101

$\underline{\quad\quad\quad 1}$ (+) Adding 1 bit (2's)

$-(0.125)_{10} = \underline{1.1110}$

2. Floating point Representation :

A positive number is represented as

$$\boxed{F = 2^{c} \cdot M}$$

where, M → Mantissa ⇒ $\boxed{1/2 \le M < 1}$

C → Exponent → either positive (or) Negative

Example: ① 2.25

$2.25 = 2^{2} \times 0.25$

$F = 2^{010} \times 0.01$

Eg② $0.125 = 2^{0} \times 0.001$

$F = 2^{000} \times 0.001$

0.25×2

0.50×2

1.00

$0.25 \rightarrow 0.01$

0.125×2

0.250×2

0.500×2

1.000

$0.125 \rightarrow 0.001$

3bit Binary

0 - 000
1 - 001
2 - 010
3 - 011
4 - 100
5 - 101
6 - 110
7 - 111

→ Negative Floating point number is represented by considering the mantissa as fixed point number.

→ The first bit of Mantissa represented the sign of the floating point number

III. Block floating point representation:

→ The combination of fixed point and floating point representations.

→ The set of signal is divided into Blocks.

→ The arithmetic operations with in the block uses Fixed Point arithmetic and only one exponent per block is stored.

Fixed point Representation	Floating point Representation
→ Fast and Inexpensive Implementation	→ Slow & expensive Implementation
→ Limited Dynamic Range	→ Large Dynamic Range
→ Round off error occur only for addition	→ Round off error can occur with both addition and multiplication.
→ Accuracy is not good	→ Accuracy is Improved

Example: ①

Find the sign magnitude 1's Complement, 2's complement for the given numbers (i) $-7/32$, (ii) $-7/8$, (iii) $+7/8$

Soln: (i) $-\frac{7}{32}$

$-\frac{7}{32} \Rightarrow -0.21875$

$-\frac{7}{32} = (-0.21875)_{10}$

$= (1.00111)_2$

0.21875
2
0.43750
2
0.87500
2
1.75000
2
1.50000
2
1.00000

Sign magnitude form = 1.00111
↓↓↓↓↓ 1's complement
1's Complement form = 1.11000
1 (+) Adding 1 → 2's complement
2's Complement form = 1.11001

(ii) $-\frac{7}{8}$

$-\frac{7}{8} = -0.875$

$-\frac{7}{8} = -(0.875)_{10} = (0.111)_2$

0.875
2
1.750
2
1.500
2
1.000

Sign magnitude form = 0.111
↓↓↓
1's Complement form = 1.000
1
2's Complement form = 1.001

(iii) $+\frac{7}{8}$

Sign magnitude form = 0.111
1's Complement form = 0.111
2's Complement form = 0.111

Problem ②

Addition of two fixed point numbers. $(0.5)_{10} + (0.125)_{10}$

$$(0.5)_{10} = (0.100)_2$$

$$(0.125)_{10} = \underline{(0.001)_2}$$

$$(0.625)_{10} = (0.101)_2$$

Addition of two Fixed point numbers causes an overflow

Eg:

$$\begin{array}{r} (0.100)_2 \\ (0.101)_2 \\ \hline (1.001)_2 \end{array}$$

$$\begin{array}{l} .001 \\ \to 0\times2^{-1} \\ \to 0\times2^{-2} \\ \to 1\times2^{-3} \end{array} \quad = 0+0+2^{-3} = (0.125)$$

$(1.001)_2 = -(0.125)_{10}$ in sign magnitude Form.

Subtraction of two Fixed point Numbers:-

Subtraction of two numbers can be easily performed easily by using two's complement representation

Subtract 0.25 from 0.5

$$\begin{array}{rcl} (0.5)_{10} & = & (0.100)_2 \\ -(0.25)_{10} & = & (1.110)_2 \\ \hline & & (10.010)_2 \end{array}$$

$$\begin{array}{r} 0.5 \\ \times 2 \\ \hline 1.0 \\ \times 2 \\ \hline 0.0 \\ \times 2 \\ \hline 0.0 \end{array} \quad (0.100)_2$$

$$\begin{array}{r} 0.25 \\ \times 2 \\ \hline 0.50 \\ \times 2 \\ \hline 1.00 \\ \times 2 \\ \hline 0.00 \end{array} \quad (0.010)_2$$

$$\begin{array}{rl} 0.010 & \to 1\text{'s} \\ 1.101 & \\ +1 & 2\text{'s} \\ \hline (1.110)_2 & \end{array}$$

Here the Carry is generated after the addition, Neglect the Carry bit to get the result in decimal.

$$(0.010)_2 = (0.25)_{10}$$

Subtract 0.5 from 0.25

$$\begin{array}{rcl} (0.25)_{10} & = & (0.010)_2 \\ -(0.5)_{10} & = & (1.100)_2 \\ \hline & & (1.110)_2 \end{array}$$

Sign magnitude Form $= (0.100)_2$

1's Complement $= (1.011)_2$

$$\begin{array}{r} +1 \\ \hline \end{array}$$

2's Complement $= (1.100)_2$

Here the Carry is not generated after addition

So the result is negative.

278 Can be represented in Floating point representation

$$278 = \frac{278 \times 1000}{1000} = \frac{278}{1000} \times 1000$$

$$F = 0.278 * 10^3$$

0.278 → Mantissa (M)

10 → Base (or) Radix (r)

3 → Exponents (e)

Example ②

$$5 = \frac{5 \times 8}{8} = \frac{5}{8} \times 8 \Rightarrow 0.625 \times 2^3$$

Mantissa (M) = 0.625

Base (or) Radix (r) = 2

Exponent (e) = 3

Example ③

Some decimal Numbers

$4.5 \rightarrow 0.5625 \times 2^3 = 0.1001 \times 2^{011}$

$1.5 \rightarrow 0.75 \times 2^1 = 0.1100 \times 2^{001}$

$6.5 \rightarrow 0.8125 \times 2^3 = 0.1100 \times 2^{011}$

$0.625 \rightarrow 0.625 \times 2^0 = 0.1010 \times 2^{000}$

Problem:

Express the fraction (-7/32) in signed magnitude and two's Complement notations Using 6-bits.

$$\left(-\frac{7}{32}\right) = 0.21875$$

$$\left(-\frac{7}{32}\right)_{10} = (0.21875)_{10} = (\underbrace{1}_{\text{-ve sign}}.001110)_2$$

Signed magnitude = $(1.001110)_2$

1's Complement = 1.110001

$$\text{2's Complement} = \frac{\begin{array}{r} 1.110001 \\ 1 \end{array}}{(1.110010)_2}$$

6 bits

$$\begin{array}{r} 0.21875 \\ 2 \\ \hline 0.43750 \\ 2 \\ \hline 0.87500 \\ 2 \\ \hline 1.75000 \\ 2 \\ \hline 1.50000 \\ 2 \\ \hline 1.00000 \\ 2 \\ \hline 0.00000 \end{array} \downarrow$$

What is meant by Floating point arithmetic ? Give Example

Floating point representation consist of Mantissa M and Exponent E. Floating point number

Binary Floating point number : $M \times 2^{E}$

Decimal Floating point number : $M \times 10^{E}$

Value of Mantissa lies between 0.5 and 1. Exponent can be positive (or) negative number 0.4882×10^{-2} 0.3815×10^{6} for Decimal.

0.1101×2^{011}, 0.11101×2^{101} For Binary

Adv: They cover wide range.

What is meant by Finite word length Effects in digital filters ?

Digital Implementation of Filter has Finite Accuracy when numbers are represented in digital Form, error are Introduced due to their finite Accuracy. these error due to generate Finite precision effects (or) Finite word length effects.

Need for signal Scaling ?

Scaling is required in the Filter Implementation to prevent overflow of value in digital Hardware. The digital hardware has finite number of bits. hence they can handle only limited range of values.

if any parameter becomes large during computation its to be scaled to prevent overflow.

Round off Noise Error :

Rounding operation is performed only on magnitude of the number. round-off noise error is Independent of type of fixed point representation.

$b \rightarrow$ before quantization

$b_u \rightarrow$ After quantization.

Max Round off Error $\left(\frac{2^{-b}-2^{-b_u}}{2}\right)$

⑪ The input to the system $y(n) = 0.999\, y(n-1) + x(n)$ is applied to an ADC. What is the power produced by the Quantisation noise at the output of filter if the Input is Quantised to

(a) 8 bit (b) 16 bits

Soln: $y(n) = 0.999\, y(n-1) + x(n)$

Take 'Z' Transform on Both sides

$$Y(z) = 0.999\, z^{-1} Y(z) + X(z)$$

$$Y(z) - 0.999\, z^{-1} Y(z) = X(z) \Rightarrow Y(z)[1 - 0.999 z^{-1}] = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 0.999 z^{-1}} \longrightarrow ①$$

Taking Inverse Z-Transform

$$h(n) = z^{-1}\left[\frac{1}{1 - 0.999 z^{-1}}\right]$$

$$\left(\because z^{-1}\left[\frac{1}{1 - a z^{-1}}\right] \Rightarrow a^n u(n) \right)$$

$$\boxed{h(n) = (0.999)^n u(n)}$$

Output Noise Power σ_{eo}^2

$$\sigma_{eo}^2 = \sigma_e^2 \sum_{n=-\infty}^{\infty} h^2(n)$$

$$= \sigma_e^2 \sum_{n=-\infty}^{\infty} \left[(0.999)^n u(n)\right]^2$$

$$\sigma_{eo}^2 = \sigma_e^2 \sum_{n=0}^{\infty} \left[(0.999)^n (1)\right]^2 \left(\because u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{otherwise} \end{cases} \right.$$

$$= \sigma_e^2 \sum_{n=0}^{\infty} (0.999)^{2n} \qquad \left(\because \sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \right.$$

$$\sigma_{eo}^2 = \sigma_e^2 \left[\frac{1}{1 - (0.999)^2}\right]$$

$$\boxed{\sigma_{eo}^2 = \sigma_e^2 [500.25]}$$

(a) 8 bit

$$\therefore \sigma_e^2 = \frac{2^{-2b}}{12}$$

$$\sigma_{eo}^2 = \frac{2^{-2b}}{12}[500.25]$$

$$\boxed{b+1 = N}$$

$b+1 = 8$

$b = 8-1 = 7 \Rightarrow \boxed{b=7}$

$$\sigma_{eo}^2 = \frac{2^{-2\times7}}{12}[500.25]$$

$$\boxed{\sigma_{eo}^2 = 2.544\times10^{-3}}$$

(b) 16 bit

$b+1 = 16 \quad \boxed{b=15}$

$$\sigma_{eo}^2 = \frac{2^{-30}}{12}[500.25]$$

$$\boxed{\sigma_{eo}^2 = 3.882\times10^{-8}}$$

② Find the Characteristics of a Limit cycle oscillation of the system (First order digital system)

$y(n) = 0.95y(n-1) + x(n)$, where $x(n) = \begin{cases} 0.875 & n=0 \\ 0 & \text{otherwise.} \end{cases}$

Also determine the deadband of the filter.

Soln: Given $a = \alpha = 0.95$, $x(0) = 0.875$

Assume that the number of bits:

$b \Rightarrow$ 4 bits + 1 sign bit

The output with Rounding value is given as

$$y(n) = x(n) + Q[0.95y(n-1)]$$

$n=0$ $\quad y(0) = x(0) + Q[0.95y(0-1)] \qquad (\because y(-1)=0$

$= 0.875 + Q[0]$

$$\boxed{y(0) = 0.875}$$

$\underline{n=1}$: $y(1) = x(1) + Q[0.95\,y(1-1)]$

$= 0 + Q[0.95\,y(0)]$

$= Q[0.95 \times 0.875]$ Rounding off

$y(1) = Q[0.83125]$

$\Rightarrow Q[0.\underbrace{1101}_{4bit}\boxed{0}\cdots]_2$ adding Quantized value (or)

0.83125
2

$\boxed{1}.66250 \rightarrow 1$
2

$\boxed{1}.32500 \rightarrow 1$
2

$\boxed{0}.65000 \rightarrow 0$
2

$\boxed{1}.30000 \rightarrow 1$
2

$\boxed{0}.60000 \rightarrow 0$
⋮

$y(1) = (0.1101)_2 \rightarrow$ Rounded value

$b=4$

$1 \times \frac{1}{2}$

$1 \times \frac{1}{4}$

$0 \times 1/8$

$1 \times 1/16$

$= \frac{1}{2} + \frac{1}{4} + \frac{1}{16}$

$\Rightarrow \frac{8+4+1}{16} \Rightarrow \frac{13}{16}$

$y(1) = Q[0.11010\cdots] = (0.1101)_2$

$$\boxed{y(1) = 0.8125}$$

$\underline{\underline{n=2}}$ $y(2) = x(2) + Q[0.95\,y(2-1)]$

$= 0 + Q(0.95 \times 0.8125)$

$= Q[0.771875]_{10}$

$=$

0.771875
$\times 2$

$\boxed{1}.543750 \rightarrow 1$
$\times 2$

$\boxed{1}.087500 \rightarrow 1$
$\times 2$

$\boxed{0}.175000 \rightarrow 0$
$\times 2$

$\boxed{0}.350000 \rightarrow 0$
$\times 2$

$\boxed{0}.700000 \rightarrow 0$

$b = 4$ bit

$\Rightarrow Q[0.\underbrace{1100}_{4\,\text{bit}}\overset{\text{adding}}{\textcircled{0}}\ldots]_2$

$y(n) = (0.1100)_2 \rightarrow$ Rounded value

$\rightarrow 1 \times \frac{1}{2} = 0.5$
$\rightarrow 1 \times \frac{1}{4} = 0.25$
$\rightarrow 0 \times \frac{1}{8} = 0$
$\rightarrow 0 \times \frac{1}{16} = 0$

0.75

$$y(2) = Q[\boxed{0}.11000\ldots]_2 = (0.1100)_2 = (0.75)_{10}$$

$$\boxed{y_2 = 0.75}$$

n=3: $y(3) = x(3) + [0.95\, y(3-1)]$

$= 0 + Q[0.95\, y(2)] \Rightarrow 0 + Q[0.95 \times 0.75]$

$= Q[0.7125]_{10}$

0.7125
$\times 2$

$\boxed{1}.4250 - 1$
$\times 2$

$\boxed{0}.8500 \rightarrow 0$
$\times 2$

$\boxed{1}.7000 \rightarrow 1$
$\times 2$

$\boxed{1}.4000 \rightarrow 1$
$\times 2$

$\boxed{0}.8000 \rightarrow 0$

$\Rightarrow Q[0.\underbrace{1011}_{4\,\text{bit}}\overset{\text{Adding}}{\textcircled{0}}]_2$

$y(n) = (0.1011)_2 \rightarrow$ Rounded value.

$\rightarrow 1 \times \frac{1}{2} = 0.5$
$\rightarrow 0 \times \frac{1}{4} = 0$
$\rightarrow 1 \times \frac{1}{8} = 0.125$
$\rightarrow 1 \times \frac{1}{16} = 0.0625$

0.6875

$$y(3) = Q[0.1011]_2 = (0.6875)_2$$

$$\boxed{Y(3) = 0.6875}$$

$\underline{n=4}:$ $y(4) = x(4) + Q[0.95\,y(4-1)]$

$= 0 + Q[0.95 \times 0.6875]$

$= Q[0.653125]_{10}$

adding

0.653125×2 $\Rightarrow Q[0.\underbrace{1010}_{4bit}\boxed{0}0]_2$

$\boxed{1}.306250 \times 2 \rightarrow 1$

$\boxed{0}.612500 \times 2 \rightarrow 0$

$\boxed{1}.225000 \times 2 \rightarrow 1$

$\boxed{0}.450000 \times 2 \rightarrow 0$

$\boxed{0}.900000 - 0$

$y(n) = (0.1010)_2 \rightarrow$ Rounded value

$1 \times \frac{1}{2} = 0.5$

$0 \times \frac{1}{4} = 0$

$1 \times \frac{1}{8} = 0.125$

$0 \times \frac{1}{16} = 0$

$\overline{0.625}$

$y(4) = (0.1010)_2 = (0.625)_{10}$

$\boxed{y(4) = 0.625}$

$\underline{n=5}$ $y(5) = x(5) + Q[0.95\,y(5-1)]$

$= 0 + Q[0.95 \times 0.625]$

$= Q[0.59375]_{10}$

Adding

0.59375×2 $Q[0.\overleftrightarrow{1001}\boxed{1}]_2 \Rightarrow [0.1010]_2$

$\boxed{1}.18750 \times 2 \rightarrow 1$

$\boxed{0}.37500 \times 2 \rightarrow 0$

$\boxed{0}.75000 \times 2 \rightarrow 0$

$\boxed{1}.50000 \times 2 \rightarrow 1$

$\boxed{1}.00000 \rightarrow ①$

$y(n) = (0.1010)_2 \rightarrow$ Rounded Value

$1 \times \frac{1}{2} = 0.5$

$0 \times 1/4 = 0$

$1 \times 1/8 = 0.125$

$0 \times 1/16 = 0$

$\overline{0.625}$

$y(5) = (0.1010)_2 = (0.625)_{10}$

$\boxed{y(5) = 0.625}$

$$y(n) = \{ \overset{y(0)}{0.875}, \overset{y(1)}{0.8125}, \overset{y(2)}{0.75}, \overset{y(3)}{0.6875}, \overset{y(4)}{0.625}, \underset{y(5)}{0.625}, 0.625 \ldots \}$$

For $n \geq 5$, the output remains constant at 0.625 causing limit cycle oscillations.

Dead Band: $\text{Dead Band} = \dfrac{\frac{1}{2} 2^{-b}}{1-|\alpha|}$

$$= \frac{\frac{1}{2} \cdot 2^{-4}}{1-0.95} \Rightarrow \frac{\frac{1}{2} \cdot \frac{1}{2^4}}{0.05} \Rightarrow \frac{1/32}{0.05}$$

$$\boxed{\text{Dead Band} = 0.625}$$

Problem 2: Explain the characteristics of limit cycle oscillations with respect to the system described by the difference equation $y(n) = Q[a y(n-1)] + x(n)$, where $y(n)$ is the output of the filter and $Q[\cdot]$ is the quantization. Assume $a = 7/8$, $x(0) = 3/4$ for $n > 0$, and choose 4 bit sign magnitude.

Soln: The difference equation of the system

$$y(n) = Q[a\, y(n-1)] + x(n)$$

$$y(n) = Q[7/8\, y(n-1)] + x(n) \qquad \because x(0) = \frac{3}{4} = 0.75$$

$$y(-1) = 0$$

$$y(n) = Q[0.875\, y(n-1)] + x(n)$$

$n = 0$ $\quad y(0) = Q[0.875\, y(0-1)] + x(0)$

$$= Q[0.875\, y(-1)] + 0.75$$

$$= Q[0] + 0.75$$

$$\boxed{y(0) = 0.75}$$

$n=1$: $y(1) = Q[0.875\,y(1-1)] + x(1)$

$= Q[0.875\,y(0)] + x(1)$

$= Q[0.875 \times 0.75] + 0$

$= Q[0.65625]_{10}$ [∵ Quantized 4 bits Including sign bit.

0.65625
2
1.31250
2
0.62500
2
1.25000
2
0.50000
2
1.00000

$Q[0.101\textcircled{0}1]_2 \Rightarrow Q[0.101]_2$

adding

$y(n) = [0.101]$ Rounded value

$1 \times 2^{-1} = 0.5$

$0 \times 2^{-2} = 0$

$1 \times 2^{-3} = 0.125$

$y(1) = 0.5+0+0.125$

$\boxed{y(1) = 0.625}$

$n=2$: $y(2) = Q[0.875\,y(2-1)] + x(2)$

$= Q[0.875\,y(1)] + 0$

$= Q[0.875 \times 0.625]$

$= Q[0.546875]_{10}$

Quantized 4 bit Including sign bits

0.546875
2
1.093750
2
0.187500
2
0.375000
2
0.750000

$Q[0.100\textcircled{0}] = Q[0.100]_2$

adding

$y(n) = [0.100]$

$1 \times 2^{-1} = 0.5$

$0 \times 2^{-2} = 0$

$0 \times 2^{-3} = 0.$

$\boxed{y(2) = 0.5}$

n=3 $y(3) = Q[0.875\, y(3-1)] + x(3)$

$= Q[0.875\, y(2)] + 0$

$= Q[0.875 \times 0.5]$

$= Q[0.4875]_{10}$ Quantized 4bit Including sign bit

Convert Decimal to Binary

$Q[0.4375)_{10} = Q(0.011\textcircled{1}]$ adding

$$\begin{array}{r} 0.011 \\ 1 \\ \hline 0.100 \\ \hline \end{array}$$

$$\begin{array}{l} 0.4375 \times 2 \\ \hline 0.8750 \times 2 \\ \hline 1.7500 \times 2 \\ \hline 1.5000 \times 2 \\ \hline 1.0000 \\ \hline \end{array}$$

$Q[0.100]_2$

$y(n) = 0.100$

$1 \times 2^{-1} = 0.5$

$0 \times 2^{-1} = 0$

$0 \times 2^{-1} = 0$

$$\boxed{y(3) = 0.5}$$

$$y(n) = \{\underset{y(0)}{0.75}, \underset{y(1)}{0.625}, \underset{y(2)}{0.5}, \underset{y(3)}{0.5} \dots\}$$

for $n \geq 3$, the output remains constant at 0.5 causing limit cycle oscillation.

Dead Band: $k = \dfrac{\frac{1}{2} 2^{-b}}{1 - |\alpha|}$

$$= \frac{\frac{1}{2} 2^{-4}}{1 - |7/8|} = \frac{0.5 \times 0.0625}{1 - 0.875}$$

$$= \frac{0.03125}{0.125}$$

$$\boxed{k = 0.25}$$

Problem:

Consider a Second order FIR filter with system Transfer function $H(z) = \dfrac{1}{1-0.85z^{-1}+0.175z^{-2}}$. Determine the effect of Quantization on poles location of the given system function is (i) Direct form (ii) Cascade form with $b = 4$ bits.

$z(1-0.5z^{-1})$

$(z-0.5)$

$(0.5z^{-1})$

Soln: (i) Direct form

$$H(z) = \frac{1}{1-\underline{0.85}z^{-1}+0.175z^{-2}}$$

Let us Quantize the Coefficient into 4 bits.

$Q[0.85]_{10}$

0.85×2

$\boxed{1}.70 \times 2$

$\boxed{1}.40 \times 2$

$\boxed{0}.80 \times 2$

$\boxed{1}.60 \times 2$

$\boxed{1}.20$

$$Q[0.85]_{10} \rightarrow [0.\underleftrightarrow{1101}1\ldots]_2$$

Truncation

discard

Quantized 4 bit $\underline{[0.1101]_2}$

0.1101

$\rightarrow 1\times2^{-1} = 2^{-1}$

$\rightarrow 1\times2^{-2} = 2^{-2}$

$\rightarrow 0\times2^{-3} = 0$

$\rightarrow 1\times2^{-4} = 2^{-4}$

$\downarrow$ (+) (add)

$$\boxed{[0.1101]_2 \Rightarrow [0.8125]_{10}}$$

$Q[0.175]_{10}$

0.175×2

$\boxed{0}.350 \times 2$

$\boxed{0}.700 \times 2$

$\boxed{1}.400 \times 2$

$\boxed{0}.800$

$$Q[0.175]_{10} \rightarrow [0.\underleftrightarrow{0010}1]_2$$

Quantized 4 bit $[0.0010]_2$

0.0010

$\rightarrow 0\times2^{-1}$

$\rightarrow 0\times2^{-2}$

$\rightarrow 1\times2^{-3}$

$\rightarrow 0\times2^{-4}$

$$\boxed{[0.0010]_2 \Rightarrow [0.125]_{10}}$$

$$\boxed{[H(z)]_q = \frac{1}{1-0.8125 z^{-1} + 0.125 z^{-2}}}$$

Ploes of the system

$$\boxed{\begin{array}{l} 0.85 \rightarrow 0.8125 \\ 0.175 \rightarrow 0.125 \end{array}}$$

(ii) Casecade form:

$$H(z) = \frac{1}{1-0.85z^{-1}+0.175z^{-2}}$$

multiply and divided by (z^2) $(z-0175)$

$$= \frac{z^2}{z^2 - \underline{0.85}\, z + 0.175}$$

$$\begin{array}{c|c} \multicolumn{2}{c}{0.175} \\ \hline 0.5 & 0.35 \end{array}$$

$$= \frac{z^2}{(z-0.5)\,(z-0.35)}$$

$$H(z) = \frac{z^2}{\cancel{z}(1-0.5z^{-1})\,\cancel{z}(1-0.35z^{-1})}$$

$$H(z) = \frac{1}{(1-\underline{0.5z^{-1}})(1-0.35z^{-1})}$$

Let us Quantized the 4 bit coefficients.

$Q[0.5]_{10}$ $\qquad Q[0.5]_{10} \rightarrow [0.1000\ldots]_2$

Quantized 4 bit $[0.1000]$

$$\begin{array}{l} 0.5 \\ \quad \times 2 \\ \hline \boxed{1}.0 \\ \quad \times 2 \\ \hline \boxed{0}.0 \\ \quad \times 2 \\ \hline \boxed{0}.0 \\ \quad \times 2 \\ \hline \boxed{0}.0 \\ \hline \end{array}$$

Decimal to Binary

0.1000

$\rightarrow 1 \times 2^{-1}$

$\rightarrow 0 \times 2^{-2}$

$\rightarrow 0 \times 2^{-3}$

$\rightarrow 0 \times 2^{-4}$

$(+)$

$$\boxed{[0.1000]_2 \rightarrow [0.5]_{10}}$$

Binary to Decimal

$Q[0.35]_{10}$

0.35×2

$\boxed{0}.70 \times 2$

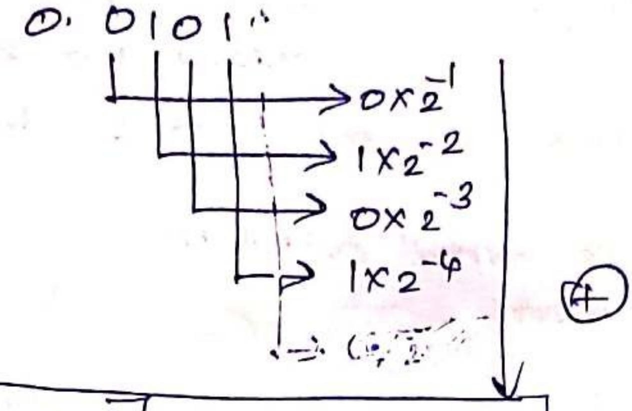
$\boxed{1}.40 \times 2$

$\boxed{0}.80 \times 2$

$\boxed{1}.60 \times 2$

$\boxed{1}.20$

$Q[0.35]_{10} \rightarrow [0.\underleftrightarrow{0101}1]_2$



$$\boxed{[0.0101]_2 \rightarrow [0.3125]_{10}}$$

The Quantized system Function is given by

$$[H(z)]_q = \frac{1}{(1-0.5z^{-1})(0.3125z^{-1})}$$

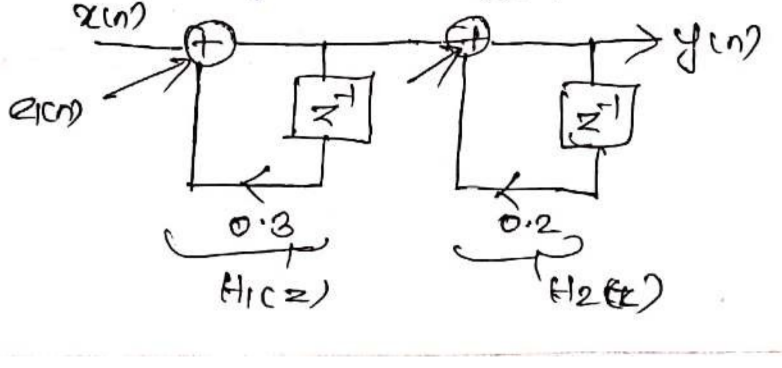
Poles Location are changed from actual position.

$$\boxed{\begin{array}{l} 0.5 \longrightarrow 0.5 \\ 0.35 \longrightarrow 0.3125 \end{array}}$$

Problem 2:

A Cascade structure has a two Individuals Section $H_1(z) = \frac{1}{1-0.3z^{-1}}$ and $H_2(z) = \frac{1}{1-0.2z^{-1}}$ Determine the overall output Noise power by product quantization noise Model.

Soln: The product Quantization Noise model For the given system.



$H(z) = H_1(z) \cdot H_2(z)$.

$$H(z) = \frac{1}{(1-0.3z^{-1})(1-0.2z^{-1})}$$

Noise Transfer Function Seen by $\boxed{\begin{array}{l} e_1(n) \text{ is } H(z) = H_1(z) \\ e_2(n) \text{ is } H_2(z) \end{array}}$

Hence overall output Noise power $\sigma_{eo}^2 = \sigma_{e1}^2 + \sigma_{e2}^2$

$\sigma_{e_1}^2 \rightarrow$ Noise power due to $e_1(n)$

$\sigma_{e_2}^2 \rightarrow$ Noise power due to $e_2(n)$

The output Noise power due to $e_1(n)$ is given by.

$$\sigma_{e_1}^2 = \sigma_e^2 \times \frac{1}{2\pi j} \oint H(z) H(z^{-1}) \frac{dz}{z}$$

$$= \sigma_e^2 \times \frac{1}{2\pi j} \oint \frac{1}{(1-0.3z^{-1})(1-0.2z^{-1})} \times \frac{1}{(1-0.3z)(1-0.2z)} \frac{dz}{z}$$

$$= \sigma_e^2 \frac{1}{2\pi j} \oint \frac{1}{z^{-1}(z-0.3) z^{-1}(z-0.2)(1-0.3z)(1-0.2z)} \frac{dz}{z}$$

$$= \sigma_e^2 \frac{1}{2\pi j} \oint \frac{1}{z^{-1}(z-0.3) z^{-1}(z-0.2)(1-0.3z)(1-0.2z)} \frac{dz}{z}$$

$$= \sigma_e^2 \frac{1}{2\pi j} \oint \frac{z}{(z-0.3)(z-0.2)(1-0.3z)(1-0.2z)} dz$$

Here the poles $z=0.3$, $z=0.2$ lies inside of unit circle

hence, $\sigma_{e_1}^2 = \sigma_e^2 \left[\text{Residue at } z=0.3 + \text{Residue at } z=0.2 \right]$

$$= \sigma_e^2 \left\{ \cancel{(z-0.3)} \frac{z}{\cancel{(z-0.3)}(z-0.2)(1-0.3z)(1-0.2z)} \Bigg|_{z=0.3} \right.$$

$$\left. + \cancel{(z-0.2)} \frac{z}{(z-0.3)\cancel{(z-0.2)}(1-0.3z)(1-0.2z)} \Bigg|_{z=0.2} \right\}$$

$$\sigma_e^2 \left[\frac{0.3}{(0.3-0.2)(1-0.3\times0.3)(1-0.2\times0.3)} + \frac{0.2}{(0.2-0.3)(1-0.3\times0.2)(0.2-0.2\times0.2} \right.$$

$$\sigma_e^2 \left[3.507 - 2.126 \right]$$

$$\boxed{\sigma_{e_1}^2 = \sigma_e^2 \, 1.291}$$

The output noise power due to $e_2(n)$ is

$$e_2(n) = \sigma_e^2 \frac{1}{2\pi j} \oint H_2(z) H_2(z^{-1}) \frac{dz}{z}$$

$$= \sigma_e^2 \frac{1}{2\pi j} \oint \frac{1}{(1-0.2z^{-1})(1-0.2z)} \frac{dz}{z}$$

$$= \sigma_e^2 \frac{1}{2\pi j} \oint \frac{1}{z^{-1}(z-0.2)(1-0.2z)} \frac{dz}{z}$$

Here the poles $\boxed{z=0.2}$ lies inside of unit circle

$1-0.2z = 0, \; 1 = 0.2z \Rightarrow z = 1/0.2 \Rightarrow \boxed{z=5}$ outside the unit circle

$$\sigma_{e_2}^2 = \sigma_e^2 \left[\text{Residue at } z=0.2 \right]$$

$$= \sigma_e^2 \left[\cancel{(z-0.2)} \frac{1}{\cancel{(z-0.2)}(1-0.2z)} \Bigg|_{z=0.2} \right]$$

$$= \sigma_e^2 \left[\frac{1}{1-0.2\times0.2} \right]$$

$$\boxed{\sigma_{e_2}^2 = \sigma_e^2 \, 1.042}$$

The over all output noise power

$$\sigma_{eo}^2 = \sigma_{e_1}^2 + \sigma_{e_2}^2$$

$$= \sigma_e^2 1.291 + \sigma_e^2 1.042 \qquad \sigma_e^2 = 2^{-2b}$$

$$\sigma_{eo}^2 = \sigma_e 4\ 2.333$$

Problem:

The output of A to D converter is applied to a digital Filter with system function $H(z) = \frac{0.5z}{z-0.5}$ Determine output noise power when the input signal is quantized 8-bits.

Given: N = 8 bit

$$H(z) = \frac{0.5z}{z-0.5}$$

$$H(z^{-1}) = \frac{0.5z^{-1}}{z^{-1}-0.5}$$

The output Noise power

$$\sigma_{eo}^2 = \sigma_e^2 \oint \frac{1}{2\pi j} \oint H(z) H(z^{-1}) \frac{dz}{z}$$

$$\sigma_{eo}^2 = \sigma_e^2 \frac{1}{2\pi j} \oint \left(\frac{0.5z}{z-0.5}\right)\left(\frac{0.5z^{-1}}{z^{-1}-0.5}\right) \frac{dz}{z}$$

$$= \sigma_e^2 \frac{1}{2\pi j} \oint \left(\frac{0.5z}{z-0.5}\right)\left(\frac{0.5z^{-1}}{z^{-1}(1-0.5z)}\right) \frac{dz}{z}$$

$$\sigma_e^2 \frac{1}{2\pi j} \oint \frac{0.5\cancel{z}}{(z-0.5)} \frac{0.5\cancel{z^{-1}}}{\cancel{z^{-1}}(1-0.5z)} \frac{dz}{\cancel{z}}$$

$$\sigma_{eo}^2 = \sigma_e^2 \frac{1}{2\pi j} \oint \frac{0.25}{(z-0.5)(1-0.5z)} dz$$

Here the poles $z = 0.5$

$1 - 0.5z = 0$

$0.5z = 1 \Rightarrow z = \frac{1}{0.5} = 2$

$\boxed{z = 0.5} \rightarrow$ Inside unit circle $\boxed{z = 2}$ outside unit circle

$$\sigma_{eo}^2 = \sigma_e^2 \left[\Sigma \text{ Residue of } \frac{0.25}{(z-0.5)(1-0.5z)} \text{ at the Poles } z=0.5,\ z=2 \right]$$

$$\sigma_{eo}^2 = \sigma_e^2 \left[\text{Residue of } \frac{0.25}{(z-0.5)(1-0.5z)} \text{ at } z=0.5 \right]$$

$$\sigma_e^2 \cancel{(z-0.5)} \left(\frac{0.25}{\cancel{(z-0.5)}(1-0.5z)} \right) \Bigg|_{z=0.5}$$

$$\sigma_e^2 = \frac{0.25}{1-0.5\times0.5} = \frac{\sigma_e^2\, 0.25}{1-0.25}$$

$$\boxed{\sigma_{eo}^2 = \sigma_e^2\ 0.33}$$

$$\sigma_{eo}^2 = \frac{2^{-2b}}{12} \times 0.33$$

$b+1 = N$

$b+1 = 8$

$b = 7$

$$= \frac{2^{-2\times7}}{12} \times 0.33$$

$$\boxed{\sigma_{eo}^2 = 5.086 \times 10^{-6}}$$

FOORMULA:

ADC: Steady state output Noise power

$$\boxed{\sigma_{eo}^2 = \sigma_e^2 \sum_{n=0}^{\infty} h^2(n)}$$

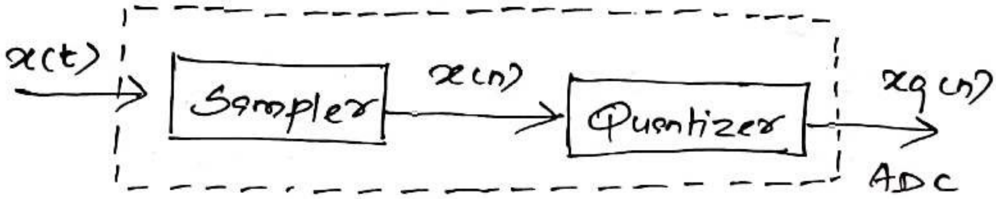
The closed Contour Integration can be evaluated using Residue theorem of Z- Transform.

$$\boxed{\sigma_{eo}^2 = \sigma_e^2 \frac{1}{2\pi j} \oint H(z) H(z^{-1}) \, dz}$$

$$\sigma_e^2 = \frac{2^{-2b}}{12}, \quad \boxed{N = b+1}$$

A/D Conversion Noise:

A digital signal processor contain a device A/D Converter that operates on the Analog i/p x(t) to produce xq(n) which is Binary sequence of 0's and 1's



The signal x(t) is sampled at regular Interval to produce a sequence x(n) of Infinite precision

Each x(n) Sample is expressed interm of a finite number of bits giving the sequence xq(n)

Difference signal $\boxed{e(n) = x_q(n) - x(n)}$ is called A/D conversion noise.

Coefficient Quantization Noise:

Definition: Quantization of Coefficients in digital Filter produce an error that is known as Coefficient Quantization error.

→ Coefficients are represented in Binary and store in registers.

→ The Coefficients must be Quantized using rounding / truncation Method.

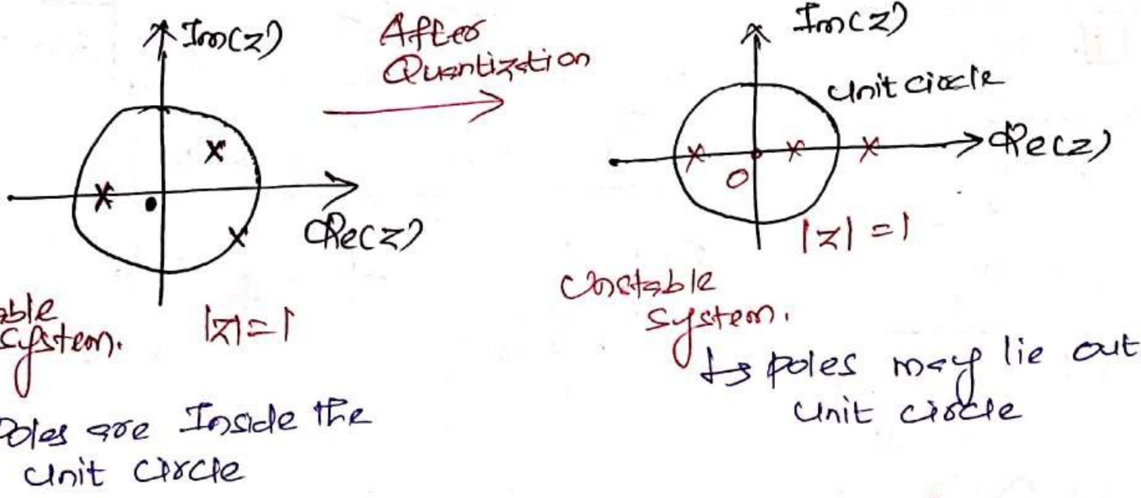
Effects of Coefficients Quantization:

→ The frequency response of the Filter deviates from the desired response of the system.

→ Filter may fail to meet the desired specifications.

→ the coefficient Quantization process Modifies the Poles - zero locations.

→ Sometime the poles location will be changed in such a way that the system may lead to instability.



Stable system.

↳ poles are Inside the unit circle

Unstable system.

↳ poles may lie outside unit circle

Method Used to reduce Coefficient Quantization:

A High order Filter as a Casecade of Second-order Section.

Problem: Find the effect of Coefficient Quantization on poles Locations for the given second order IIR filter Using Direct Form - I.

$$H(z) = \frac{1}{(1-0.5z^{-1})(1-0.45z^{-1})}$$

Soln:

Take 3 bits for Truncation:

$$H(z) = \frac{1}{(1-0.5z^{-1})(1-0.45z^{-1})}$$

$$= \frac{1}{1-0.45z^{-1}-0.5z^{-1}+0.225z^{-2}}$$

$$H(z) = \frac{1}{1-0.95z^{-1}+0.225z^{-2}}$$

Coefficient Quantization:

$$H(z) = \frac{1}{1-Q[0.95z^{-1}]+Q[0.225z^{-2}]}$$

$Q[0.95]_{10}$

0.95_{2}

$\boxed{1}.90_{2}$

$\boxed{1}.80_{2}$

$\boxed{1}.60_{2}$

$\boxed{1}.20.$

$$Q[0.95]_2 \Rightarrow Q[0.\underbrace{111}1\cdots]_2 \xrightarrow{} \text{discard}$$

$$= (0.111)_2$$ 3bit Truncation.

$$\boxed{Q[0.95]_{10} = (0.875)_{10}}$$

$$0.111 \quad \to 1\times2^{-1},\ 1\times2^{-2},\ 1\times2^{-3}$$

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8} = 0.875$$

$Q[0.225]_{10}$

$0.22\overset{2}{}$

$\times 2$

$\boxed{0}\,45\,0$

$\times 2$

$\boxed{0}.\,900$

$\times 2$

$\boxed{1}\,800$

$\times 2$

$\boxed{1}\,600$

$\times 2$

$\boxed{1}\,200$

$$Q[0.225]_{10} \Rightarrow [0.\underleftrightarrow{001}\underbrace{11\cdots}_{\text{discard}}]_2$$

$$= [0.001]_2$$

0.001

$0 \times 1/2$

$0 \times 1/4$

$1 \times \frac{1}{8}$

$$\Rightarrow 0 + 0 + \frac{1}{8} = 0.125$$

$$Q[0.225]_{10} = (0.125)_{10}$$

$$H(z) = \frac{1}{1 - 0.875 z^{-1} + 0.125 z^{-2}}$$

Poles location are changed to

$$\boxed{\begin{aligned} 0.95 &\rightarrow 0.875 \\ 0.225 &\rightarrow 0.125 \end{aligned}}$$

Quantization of Filter Coefficients in Cascade Method:

$$H(z) = \frac{1}{(1 - 0.5z^{-1})(1 - 0.45z^{-1})}$$

No. of bits = 3 : Truncation Method

$Q[0.5]$

0.5

$\times 2$

$\boxed{1}.\,0$

$$Q[0.5]_{10} = Q[0.100]_2$$

$$\boxed{Q[0.5] = 0.5}$$

$Q[0.45]$

0.45

$\times 2$

$\boxed{0}.\,90$

$\times 2$

$\boxed{1}.\,80$

$\times 2$

$\boxed{1}.\,60$

$\times 2$

$\boxed{1}.\,20$

$$Q[0.45]_{10} = Q[0.011\boxed{1}]_2$$

0.011

$0 \times 1/2 = 0$

$1 \times 1/4 = 1/4$

$1 \times 1/8 = 1/8$

$$\boxed{Q[0.45] = 0.375}$$

FINITE WORD LENGTH EFFECTS:

→ Digital system the numbers and Coefficients are Stored in Finite length registers.

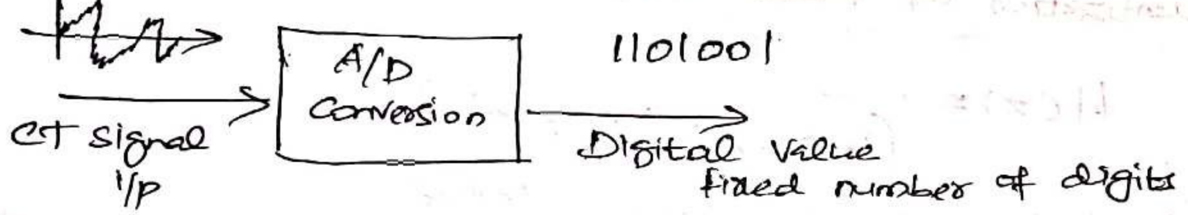
The numbers are Quantized by truncation (or) Rounding off.

Effects | Errors due to Quantization:

(i) Input Quantization Error
(ii) Product Quantization Error
(iii) Coefficient Quantization Error
(iv) Round off Noise
(v) Limit cycle oscillations.

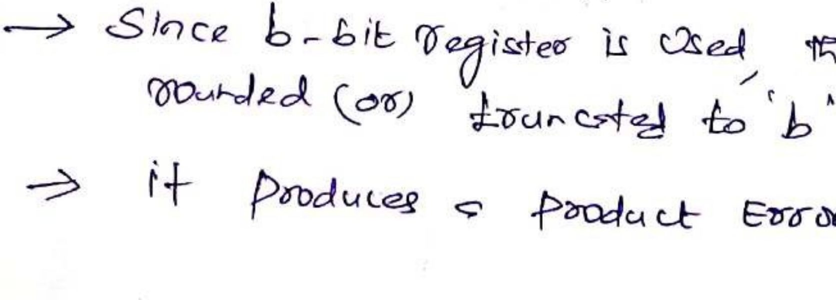
(i) Input Quantization Error:

→ Input Quantization error occurs when the Continuous time signal is Converted into digital value in A/D conversion process.



(ii) Product Quantization Error:

→ Product Quantization Error occurs at the o/p of the multiplier.



→ Since b-bit register is used the multiplier o/p must be rounded (or) truncated to 'b' bits

→ it produces a product Error.

(iii) Coefficient Quantization Error:

→ This error arises due to process of quantizing the filter coefficients.

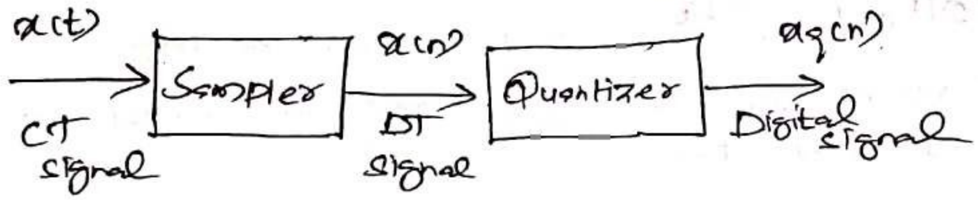
→ If the filter coefficients are quantized the frequency response of the resulting filter may differ from desired response

(iv) Limit cycle oscillation:

→ Its a Low level oscillation which arises due to the non-linearity associated with rounding off the Internal filter coefficients

(i) Zero input Limit cycle oscillation
(ii) Overflow limit cycle oscillation

Quantization:



Quantizer: → To represent the numeric equivalent of each sample of DT signal $x(n)$ as a finite number of bits $x_q(n)$.

Quantization: Process of digitizing the range of signal

Quantization Noise: Difference between $x_q(n)$ & $x(n)$ [unquantized & Quantized]

$$\boxed{e(n) = x_q(n) - x(n)}$$

$b+1$ → No. of bits including sign bit

Quantized step size $$\boxed{q = \frac{2}{2^{b+1}} = 2^{-b}}$$

Method of Quantization:

(i) Truncation
(ii) Rounding.

(i) Truncation: Is the process of Discarding all bit less Significant than the desired LSB which is retained.

Ex: Truncate the 8 bit Binary number to 4 bits

$$1.\underline{0011}\,\underbrace{0110}_{\text{discard}} \longrightarrow 1.0011$$

$$0.\underline{0100}1001 \longrightarrow 0.0100$$

$$8\text{ bits} \xrightarrow[\text{Truncation}]{} 4\text{ bits}$$

Rounding: Its process of choosing the rounded value (b-bit number) which is the nearest to the original value

Ex:

$$0.1101101 \xrightarrow[4\text{bit}]{7\text{bit to}} 0.1110 \text{ (or) } 0.1100$$

Rounding up (or) down will have negligible effect on accuracy of computation.

Overflow Limit cycle oscillation:

→ When the sum of Fixed point numbers exceeds the size of the register in the digital system. the overflow error occur in the output

→ This overflow error causes the Filter to oscillates between the maximum and minimum amplitudes.

→ Limit cycle known as overflow oscillation due to Addition

Ex: Consider a 4 bit digital system

$$x_1(n) = 1\ 1\ 1\ 0$$

$$x_2(n) = 1\ 0\ 1\ 0\ (+)$$

$$\underset{\text{overflow error}}{\boxed{1}}\ 1\ 0\ 0\ 0$$

ii) Let consider two postive numbers.

$$\begin{array}{llll} x_1(n) = & 0.111 & \xrightarrow{\text{Decimal value}} & 0.875 \\ x_2(n) = & 0.110 \ (+) & & 0.750 \ (+) \\ \hline & 1.101 & & \underset{\hookrightarrow -ve}{①}625 \Rightarrow -0.625 \end{array}$$

i/p → +ve, o/p → -ve

Methods Used to prevent overflow Error:

(i) Saturation Arithmetic
(ii) signal scaling.

(i) Saturation Arithmetic: when an overflow is sensed, the Sum of adder is set equal to the maximum value

Disadvantage: Undesirable signal distortion due to Non-Linearity adder.

(ii) Signal scaling: Reduce the amount of non-linear distortion, the signal is Scaled up to a certain level

→ Scale factor So Introduced between the i/p x(n) and the adder to prevent overflow at adder.

Derive the Expression for Quantization Noise power

Input Noise power:

Error signal e(n) has the following properties.

(i) The error signal e(n) is a stationary Random process
(ii) The error sequence is uncorrelated with x(n)
(iii) The error is a white noise process with uniform amplitude probability distribution. over the range of Quantization error.

The Probability Density Function for Error signal

$$P(e) = \frac{1}{2^{-b}} ; \quad \frac{-2^{-b}}{2}, \le e \le \frac{2^{-b}}{2}$$

The Power of the quantization noise is equal to the Variance of the error

The power of the quantization noise is equal to the Variance of the error signal is given by

$$\sigma_e^2 = E\left[e^2(n)\right] - \left[E(e(n))\right]^2$$

e(n) is uncorrelated the mean value

$$E[e(n)] = 0$$

$$\sigma_e^2 = E\left[e^2(n)\right]$$

$$\boxed{\sigma_e^2 = \int e^2 \, p(e) de}$$

$$= \int_{-2^{-b}/2}^{2^{-b}/2} e^2 \left(\frac{1}{2^{-b}}\right) de \Rightarrow \frac{1}{2^{-b}}\left[\frac{e^3}{3}\right]_{-2^{-b}/2}^{2^{-b}/2}$$

$$= \frac{1}{2^{-b} \times 3}\left[\left(\frac{2^{-b}}{2}\right)^3 - \left(\frac{-2^{-b}}{2}\right)^3\right]$$

$$= \frac{1}{2^{-b} \times 3}\left(\frac{2^{-b} + 2^{-b}}{8}\right) \Rightarrow \frac{2 \times 2^{-3b}}{24 \times 2^{-b}}$$

$$\boxed{\sigma_e^2 = \frac{2^{-2b}}{12}}$$

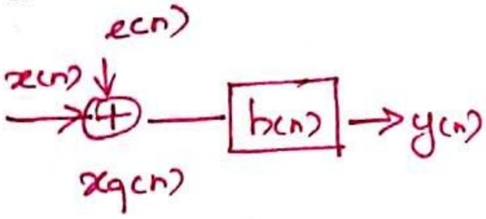
(ii) Steady state output Noise power:

After converting the CT signal into digital signal that quantized signal is applied to a digital system with system function H(z).

After quantization we have quantization noise power σ_e^2 as input noise power of Digital system is given by

$\sigma_{e0}^2 = \sigma_e^2$ system power gain

$$\sigma_{e0}^2 = \sigma_e^{-2} \sum_{n=0}^{\infty} h^2(n) \longrightarrow ①$$



$$Z[h(n)] = \sum_{n=0}^{\infty} h(n) z^{-n}$$

$$Z[h^2(n)] = \sum_{n=0}^{\infty} h^2(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} h(n) h(n) z^{-n} \longrightarrow ②$$

Take Inverse z Transform

$$h(n) = \frac{1}{2\pi j} \oint H(z) z^{n-1} dz \longrightarrow ③$$

Sub eqn ③ in eqn ②

$$Z[h^2(n)] = \sum_{n=0}^{\infty} h(n) \left[\frac{1}{2\pi j} \oint H(z) z^{n-1} dz \right] z^{-n}$$

$$Z[h^2(n)] = \frac{1}{2\pi j} \oint H(z) \sum_{n=0}^{\infty} h(n) z^{n-1} z^{-n} dz$$

$$Z[h^2(n)] = \frac{1}{2\pi j} \oint H(z) \sum_{n=0}^{\infty} h(n) \frac{dz}{z}$$

$$\sum_{n=0}^{\infty} h^2(n) z^{-n} = \frac{1}{2\pi j} \oint H(z) \sum_{n=0}^{\infty} \frac{dz}{z}$$

$$\sum_{n=0}^{\infty} h^2(n) = \frac{1}{2\pi j} \oint H(z) \sum_{n=0}^{\infty} h(n) z^{n} \frac{dz}{z}$$

$$= \frac{1}{2\pi j} \oint H(z) \sum_{n=0}^{\infty} h(n) (z^{-1})^{-n} \frac{dz}{z}$$

$$\sum_{n=0}^{\infty} h^2(n) = \frac{1}{2\pi j} \oint H(z) H(z^{-1}) \frac{dz}{z} \longrightarrow ④$$

Sub ④ in eqn ①

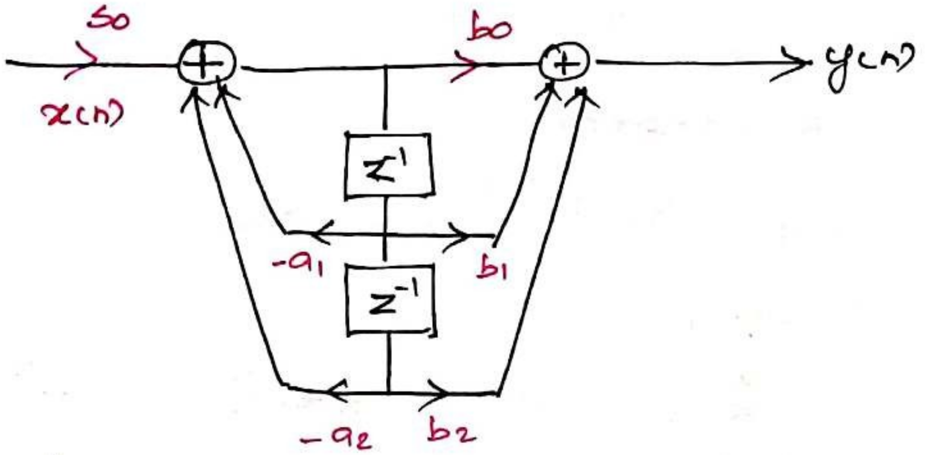
$$\boxed{\sigma_{e0}^2 = \sigma_e^2 \frac{1}{2\pi j} \oint H(z) H(z^{-1}) \frac{dz}{z}}$$

SIGNAL SCALING:

In order to limit the amount of non-linear distortion its important to scale the input signal and the unit sample response between the input and any internal summing node in the system.

that overflow become rare event

Let us consider IInd order FIR filter shown in Fig.



A scaling factor so is introduced between input x(n) and the adder I to prevent the overflow at the output adder I. the scaling factor is expressed as,

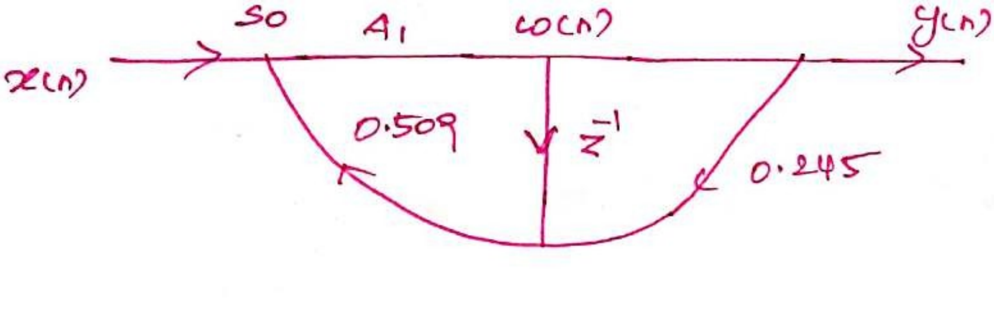
$$So^2 = \frac{1}{1/2\pi j \oint \frac{1}{D(z)D(z^{-1})} \frac{dz}{z}}$$

where,

$$I = \frac{1}{2\pi j} \oint \frac{1}{D(z)D(z^{-1})} \frac{dz}{z}$$

$$So^2 = \frac{1}{I} \qquad \boxed{So = \sqrt{1/I}}$$

Problem: for the digital Network shown in fig. Find H(z) and scale Factor. So to avoid overflow regarding A$_1$



Soln:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{0.245 z^{-1}}{1+0.509 z^{-1}}$$

Scaling factor $$S_0^2 = \frac{1}{\frac{1}{2\pi j} \oint \frac{1}{D(z) D(z^{-1})} \frac{dz}{z}}$$

$D(z) = 1+0.509 z^{-1}$

$D(z^{-1}) = 1+0.509 z$

$$S_0^2 = \frac{1}{2\pi j} \oint_c \frac{z^{-1} dz}{D(z) D(z^{-1})}$$

$$= \frac{1}{2\pi j} \oint_c \frac{z^{-1} dz}{(1+0.509 z^{-1})(1+0.509 z)}$$

$$= \frac{1}{2\pi j} \oint_c \frac{z^{-1} dz}{z^{-1}(z+0.509)+(1+0.509 z)}$$

find the residue of $\frac{z^{-1}}{(1+0.509 z^{-1})(1+0.509 z)}$ the residue due to pole $z = \frac{1}{0.509}$ is zero [exceed the unit circle] the residue due to pole $z = 0.509$ is given by.

$$\cancel{(z+0.509)} \times \frac{1}{\cancel{(z+0.509)}(1+0.509 z)} \Bigg|_{z=-0.509}$$

$$= \frac{1}{1+0.509 \times -0.509} = \frac{1}{1.3496}$$

$S_0^2 = 0.740$

$S = \sqrt{0.740}$

Scaling Factor $\boxed{S = 0.86}$